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Geology. — *"Concerning the Existence of Elevation Craters in the Dutch East Indies, as supposed by some geologists."* By L. RUTTEN.

(Communicated at the meeting of February 28, 1925).

It is well-known that in the controversy between the adherents to the Accumulation-theory (SCROPE) and the Elevation-theory of craters (VON BUCH) F. JUNGHUHN [1]) has sided with SCROPE, in the light of his own experience of volcanoes in Java. We also know that the accumulation-theory was paramount during many decennia, but that gradually a greater activity was again assigned to the magma. It has been conclusively demonstrated that the magma is possessed of such intrinsic energy, that under certain circumstances it can force its way through the earth's crust even there where no fissures existed before. Some authors, however, went the length of assigning an elevating power to the magma, and again accepted VON BUCH's theory of the existence of "Elevation-Craters". Strange to say, in the same Dutch Indies, which had formerly furnished so many arguments for the accumulation-theory, geologists now supposed to discover instances of Elevation-Craters. [2]) One of these instances has even found its way in a great Textbook of Geology [3]).

It is not my object to ascertain whether real Elevation-Craters have been found in other regions. My intention is only to show that the instances adduced from the archipelago do not afford any evidence whatever in favour of "Elevation-Craters". I feel the more prompted to do so as, to my knowledge, the views of VOLZ and RECK (l.c.) have never been contradicted, while moreover VOLZ's conception has been more or less authorized by KAYSER in his Textbook.

W. VOLZ holds that the porphyrites, which according to him and to J. HOOZE [4]) constitute the nucleus of the northern part of Pulo Laut, to the east of Borneo, are younger than the surrounding sediments, and that the porphyritic magma in forcing itself upwards has raised the latter so as to form a dome. To maintain this view he had in the first place to alter HOOZE'S representations of the geology of the island, since according to HOOZE (l.c.) in the island of P. Laut an eocene formation of sandstones, shales, and pitchcoal of about 100 m. thickness superimposes older porphyrite and diabase, while farther southward, in the centre of the island, HOOZE also recognizes diorite and serpentine. VOLZ, on the other hand, maintains that the porphyrites in the north are posteocene. True, HOOZE has not produced cogent evidence for the asserted older age of the porphyrites. This assertion is quite conceivable, however, when we reflect that in HOOZE'S time it was universally accepted that a formation, consisting of porphyrites and diabases, and to which also diorite and serpentine are added, must needs be pretertiary. The only passage in HOOZE'S work that may be called evidential, is found on p. 406: „where the diabase is in contact with the tertiary coal-formation, we are often confronted with a conglomerate-, and breccia-formation (Pamantjingan. Selaru etc.), which, therefore, is usually to be considered as the basis of this coal-formation". VOLZ has justly remarked, that these conglomerates and these breccia of volcanic material might as well be contact breccia (l.c. p. 360). It must be admitted, therefore, that HOOZE has not proved conclusively that the tertiary deposits are younger than the igneous rocks. But neither has the reverse been proved by VOLZ, in spite of his three arguments in favour of the higher antiquity of the sediments (l.c. p. 359).

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[1]) F. JUNGHUHN, Java, 1854. III.

[2]) W. VOLZ, Neues Jahrb. f. Mineralógie etc. Beil. Bd. XX. 1905. p. 354—364.
H. RECK, Zeitschr. d. Deutsch. geol. Ges. 1910. Monatsber. p. 292—318.

[3]) E. KAYSER, Lehrb. der Geologie, II. 6e ed. 1921, p. 6.: „In gleicher Weise hat W. VOLZ auf der Insel Pulo Laut bei Borneo einen Fall von Aufbiegung der Schichten in Folge eines grossen Magma-Ergusses beobachtet".

[4]) J. A. HOOZE, Jaarb. Mijnw. 1888. Techn. Adm. Gedeelte, p. 337—429.

In the first place he found "im Urwald einen Contact von Porphyrit und Thon; der Thon war direct am Contact auf geringe Entfernungen gefrittet". Now it is certain that for a contact between a very thick diabase- and a porphyrite-mass of 5 km. broad, this contact-metamorphism must be considered very poor. Further contact-metamorphism is absolutely unknown: the composition of the coal is invariably that of the normal Borneo eocene coal; the pelitic rocks possess the normal consistency of old tertiary shales; the sandstones display no trace of contact metamorphism and the Foraminifera-limes and the Orbitoides previously examined by the present writer had changed only very little diagenetically [1]). However, every one, acquainted with disclosures in primeval forests, knows how little a supposed contact between porphyrite and burnt clay signifies. In a primeval forest with intense, partly lateritic weathering, many things can gradually assume the appearance of burnt clay. Indeed, VOLZ himself seems not to attach great value to this argument.

In the second place volcanic tuffs near Sigam are lying rather level, while in the vicinity eocene layers show dips of 18°—30°; lastly, and this is considered by VOLZ to be the most important argument — in the valley of the Sg. Semblimbingam horizontal tuffs occur between upheaved eocene layers in the west, and porphyrite in the east. It stands to reason — as also appears from VOLZ's picture — that we have not to do here with a continuous row of disclosures. It now seems to me that these observations cannot prove anything, since VOLZ has not shown that the deposits, supposed to be tuffs, are not detritus of older volcanic rocks which have been very recently redeposited. This possibility deserves no doubt consideration because deposits in a valley are concerned. It may be difficult to demonstrate this, but unless this proof be brought forward the youth of the porphyrites cannot be considered as established.

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[1]) L. RUTTEN, Jaarb. Mijnw. 1914. Verh. 2. 1915. p. 74—77.

So, for the present the age of the P. Laut porphyrites remains doubtful, although there seem to be indications of their being younger than eocene. But we know that there are many difficulties in establishing the age of effusive rocks in the tropics, notably in South Borneo. The literature confirms this. The effusive rocks in the neighbourhood of Pengaron were supposed by VERBEEK to be decidedly posteocene [1]; HOOZE considered them as cretaceous porphyrites [2] while Kemmerling again referred them to the posteocene, with some reserve, though [3].

In addition I wish to say that a sandstone from Kota Baru in P. Laut, (near the lighthouse) collected by myself some ten years ago, has been entirely (or nearly so) made up of quartz-grains and finely granular siliceous rocks, while also HOOZE (l. c. p. 376) records that the sandstones from the eocene of P. Laut are often of a pure white. This would not imply a higher age of the porphyrites, inasmuch as in the sandstones the detritus of the porphyrites is lacking altogether. There is, however, a possibility that this detritus occurs only in the lowermost parts of the eocene formation.

In the second place VOLZ has failed to show that the entire nucleus of northern P. Laut is composed of diabases and porphyrite, that we have really to do here with an extensive, connected eruptive mass. With the exception of his discovery of diabase-porphyrite at the highest top of the island, HOOZE's observations have been restricted to the coastal zône. VOLZ, it is true, has traversed the whole island ("Das Gebirge ist mit dichtem Urwalde bedeckt" p. 356), but there is no evidence whatever of his having then gathered enough geological data to warrant the conclusion that the entire nucleus of northern P. Laut is one continuous eruptive mass. Given the fact that in the south of this island quartz-diorite and serpentine occur, which elsewhere in S. Borneo are often associated with schists, I do not think it at all impossible that these rocks occur also in the centre of the northern part of the island, and that the material of the sandstones from the eocene has been supplied by the schists. It would probably put any engineer in P. Laut to very little trouble if he should devote his off-days to some excursions in the principal valleys on the west coast, in which the very boulders would supply important hints as to the structure of the rocks in the interior.

So, when propounding his highly important hypothesis: "Die Insel Pulo Laut ... (ein) Beispiel einer Hebung durch einen Massenerguss",

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[1] R. D. M. VERBEEK, Jaarb. Mijnw. 1875, I. p. 3—130.

[2] J. A. HOOZE, Jaarb. Mijnw. 1893, p. 1—431.

[3] G. KEMMERLING, Tijdschr. Kon. Ned. Aardr. Gen. 1915. p. 717—774.

VOLZ has omitted to prove two very weighty matters: 1⁰ that the eruptiva are indeed posteocene; 2⁰. that the nucleus of P. Laut is indeed one large continuous mass of eruptiva [1]).

But even if future investigation should put VOLZ in the right on these two points, Pulu Laut would never afford *conclusive evidence* for the existence of Elevation-Craters or of related phenomena. It may reasonably be *believed* that the eruptiva of P. Laut have pushed up a portion of the earth's crust, but every effort to prove this would be hampered by the fact that P. Laut lies in the region of S.-E. and E. Borneo, where folding is a normal phenomenon, and where the main trend of the folding movements coincides with the direction of the eocene strata in the island, so that it might always be argued that the upheaval of the strata results from normal folding [2]), which would, indeed be the most plausible view. VOLZ slurs over this difficulty when he says: "Zwar wurde Indonesien nach dem Eozän so wie an der Wende von Tertiär und Dilivium von Faltungen heimgesucht (deren Wesen und Wirkung aber für Borneo so gut wie unbekannt ist), aber ich glaube nicht, dass man die oben beschriebenen, eigenartigen aber geringfügigen Dislokationen auf eine Faltung zurückführen kann" (p. 363). Now, VOLZ could have known that as early as 1905 records were on hand, from which various information regarding the folding in Borneo could be obtained. Not only had C. SCHMIDT [3]) pointed out, in accordance with modern conceptions, that there are in N. W. Borneo normal, tertiary folding-mountains, but since long folding processes in the tertiary had also been established for S. Borneo and E. Borneo [4]). It might still be assumed that in some localities those folding processes were owing to the action of igneous rocks — a conception which, from its very nature, may occasionally be met with in the oldest literature — but in many other localities no facts could be discovered that implied the co-operation of igneous rocks in the folding process. In some places the folds were very sharp, in others weak, while the anticlines displayed the semi-domelike shape, which is known from P. Laut (i. a. at the Berau-river. HOOZE l. c.). Besides in VOLZ'S time already the literature contained records concerning the general tectonic trend in S. E. and E. Borneo N.-E. to N.-N.-E., which consorts with the main trend of the coal-layers in P. Laut. These data have of course been largely accentuated by subsequent researches [1]), so that we feel sure nowadays that a perfectly normal tertiary range of folding mountains runs all along the east coast of Borneo, certainly as far as the Dutch boundary in the north.

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[1]) While I was correcting the proofsheets E. GÖLLNER's publication (Jaarb. v. h. Mijnw. 50, Verh. 1, 1921, 1924) came to my notice. It seems to show that VOLZ's view is erroneous, and that porphyrites and the diabases of P. Laut are pre-eocene.

[2]) It should be noted that the interpretations given by HOOZE of the causes of the upheaval of the eocene strata in P. Laut are very primitive, and slightly contradictory inter se: VOLZ refers to these conceptions at large — which, by the way, were completely out-of-date even in his time.

[3]) C. SCHMIDT, Bull. Soc. géol. de France (IV). 1. 1901. p. 266—267.

[4]) See also for S.-E. Borneo VERBEEK l.c. 1875, HOOZE l.c. 1893; for Kutei HOOZE Jaarb. Mijnw. 1887 Tech. Adm. Ged. and 1888 Techn. Adm. Gedeelte; for Berau HOOZE Jaarb. Mijnw. 1886 Tech. Adm. Ged.

Consequently VOLZ'S *pronouncement that P. Laut is an instance of upheaval through volcanic eruption is unproved, unprovable and contrary to all reasonable expectation.*

H. RECK (l.c.) thought he could point out several Elevation-Craters in Java, but in doing so he committed even more serious errors than VOLZ. In the first place he refers to the G. Lima to the east of Patjitan: In the middle of VERBEEK and FENNEMA's map [2]) there occurs a diatrema: "zwischen ihn und das jüngere Miocän schiebt sich ein Kranz älteren Miocäns, das den Kern von allen Seiten umschliesst, ein Lagerungsverhältniss, das nur durch die vulkanische emportreibende Kraft des zentralen Kernes erklärt werden kann" (p. 308). Obviously RECK has neglected to study properly the text accompanying VERBEEK and FENNEMA's map, else he would have seen, that the nucleus of the G. Patjitan is referred to the "old Andesites", that the inner rim of miocene belongs to the m^1, and is, therefore, younger than the nucleus, and that the outer rim of m^3 is of course still younger; consequently it is clear that according to VERBEEK and FENNEMA the older andesite can never have pushed up the younger miocene (m^1 and m^3). It is very well possible, though, that the true condition differs from the one illustrated on the map, but even then the existence of "Elevation-Craters" cannot reasonably be assumed. At all events for the present the G. Lima near Patjitan does not furnish any evidence *in favour* of the Elevation-Crater theory. Furthermore RECK has endeavoured to prove that the G. Butak in North-Rembang is a typical Elevation-Crater (because the tertiary strata dip away radially from the mountain) and that the present elevated position of the tertiary mass between G. Putjak and Lasem (Ngargopuro) resulted from a volcanic upheaval, "da eine so starke tektonische Aufwölbung in dieser Gegend gänzlich überraschend wäre" (p. 310).

Here again RECK has disregarded without bringing forward fresh evidence VERBEEK and FENNEMA's view that the three andesite-massifs belong to the "old andesites" and are older than the surrounding m^3. But even when admitting what came true later on [3]), viz. that the volcanic masses of North Rembang are younger than most sediments surrounding them, it will not do to use RECK's arguments in favour of Elevation-

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[1]) See i.a. H. JEZLER, Z.f. prakt. Geologie. 1916, p. 77—85, 113—125; L. VAN ES. Jaarb. Mijnw. 1917. Verh. 2. p. 5—143; L. KROL, Jaarb. Mijnw. 1918. Verh. 1. 1920, p. 281—367.

[2]) R. VERBEEK et R. FENNEMA, Géologie de Java et Madoura, 1896.

[3]) C. 'T HOEN. Jaarb. Mijnw. 1916. Verhandel. 2. p. 202—254.

Craters. Indeed, from VERBEEK and FENNEMA's map we knew already that North-Rembang is situated in a true folding region. Now it is not at all strange that when a complex of sediments including a diatrema are folded collectively, the sedimentary strata will, after the folding process, usually dip away from the more rigid eruptive masses. It might also be conceived — without assigning elevating power to the magma — that in certain cases an eruptive mass, which forces itself upwards just when the folding is taking place, goes to the highest parts of the anticlines, i. e. to their dome-shaped elevations. Neither is the fact that part of the tertiary sediments lies on a rather high topographical level, in any way indicative of volcanic upheaval, as we knew already from the profile of VERBEEK and FENNEMA that considerable amplitudes occurred during the folding of the tertiary, and also that since that folding intense denudation has taken place. The actual occurrence of tertiary masses on such a great height may be readily explained if we assume that there the tertiary has been preserved between two more resistant masses of volcanic rocks, whereas elsewhere it has, at such a level, been long since denuded.

Meantime our knowledge of the geology of the region has been greatly clarified by 'T HOEN's profiles (l.c.) so that RECK's conceptions are still less excusable. It now appears that the G. Butak andesite has pushed itself upon an anticline, but not on its highest part, because from the W. the sediments dip away under the G. Butak. It also appears, on the contrary that the G. Lasem has come to eruption above a synclinal region and that on all sides the sediments dip away under it, while lastly the G. Putjak has come to eruption at the side of an anticline. Further it appears that, besides the half-dome in which the G. Butak-andesite has broken through, two other beautifully formed domes occur in the territory (that of Ngandang-Lodan and that of G. Banju-Uja) into which, however, no andesite masses have broken through. He who is familiar with the geology of Java, also knows that further southward in Rembang there are still a number of dome-shaped elevations, on which the oil-boring territories are situated. The domes, therefore, have originated through normal orogenic movements and only once — the G. Butak — has an andesite-mass broken through.

The conclusion, therefore, is warranted that neither the observations made by VOLZ, *nor the interpretations brought forward by* RECK *tend to prove anything for the existence of Elevation-Craters in the Dutch East Indies.*

Utrecht, Jan. 14, 1925.

Mathematics. — "*A Representation of the Plane Pencils in R_3 on the Conics of a Plane*". By Prof. J. WOLFF. (Communicated by Prof. JAN DE VRIES).

(Communicated at the meeting of March 28, 1925).

This paper owes its origin to a talk with Prof. JAN DE VRIES, where the latter raised the question whether such a representation can be easily brought about.

1. We assume in R_3 four non-coplanar points A_1, A_2, A_3, A_4, and a plane α which does not contain any of the four points A_i. The point of intersection of α with the line $A_i\,A_j$ we call B_{ij}, the intersection of the plane $A_j\,A_k\,A_l = \alpha_i$ with α we call b_i. Further we choose a point A in α which does not lie on any line $B_{ij}\,B_{kl}$ nor on any b_i.

The ∞^5 quadratic surfaces through the four points A_i we call O^2. If (P, π) is a plane pencil (P and π are resp. the point and the plane of the pencil), we associate to it the O^2 for which P is the pole of α and A the pole of π. O^2 cuts α in a conic k^2, the image of (P, π).

2. Any non-degenerate conic k^2 of α is the image of one plane pencil. For one O^2 passes through k^2; otherwise there would pass through it a pencil O^2 of which the base curve would consist of k^2 and another plane conic which would pass through A_i. Further α has one pole P outside α with regard to O^2 because k^2 is non-degenerate and for the same reason A has one polar plane π through P. The plane pencil (P, π) has k^2 as image.

3. If k^2 consists of two straight lines k and k' with point of intersection P outside A, and if the O^2 through this k^2 is not a cone, P is the (only) pole of α relative to O^2, and A has one polar plane π, (through P), hence k^2 is the image of one plane pencil (P, π).

If O^2 is a cone with vertex in α, A has one polar plane π with regard to O^2 and α has one polar line λ unless O^2 consists of two planes which cut each other along a line in α. Apart from the latter case k^2 is the image of the ∞^1 plane pencils (P, π) where P describes the line λ. There are 7 O^2 which each consist of two planes with intersection in α, to wit 4 O^2 each consisting of a plane α_i and the plane through A_i and the intersection b_i, besides the 3 O^2 of which each consists of the planes $A_i\,A_j\,B_{kl}$ and $A_k\,A_l\,B_{ij}$. Now A has one polar plane π relative to each of the 7 pairs of planes, hence each of the 7 double lines b_i and $B^{ij} B_{kl}$ is the image of the ∞^2 pencils that fill a plane π. The 7 straight lines are the sides and the diagonals of a quadrilateral.

If k^2 consists of 2 straight lines k and k' through A and if the O^2 through this k^2 is not a cone, only the pencil (A, α) is associated to k^2. If O^2 is a cone K^2, α has a polar line λ relative to K^2; if P is a point on λ and π an arbitrary plane through P, k^2 is the image of the plane pencil (P, π), for A is the pole of π and P is the pole of α relative to K^2. These singular pairs of lines k, k' form an involution, the associated K^2 form the pencil of cones with vertex A (through the A_i); hence the λ form a quadratic cone Λ^2.

Summarizing: *all the singular k^2, images of an infinite number of plane pencils, are degenerate. They are*:

1. *the ∞^3 intersections of the cones O^2 with vertices in α outside A. Apart from 7 double lines, intersections of pairs of planes, each of them is the image of ∞^1 plane pencil (P, π) in a plane π where P describes a straight line. The 7 double lines, sides and diagonals of a quadrilateral, are each the image of the ∞^2 pencils in a plane π.*
2. *The ∞^1 intersections of the cones K^2 with vertex A. Each is the image of the ∞^3 plane pencils of which the carrier P describes a line λ. These λ form a quadratic cone Λ^2 with vertex A.*

4. Now we shall try to find the singular plane pencils (to which an infinite number of k^2 are associated). We begin with the case $P \neq A$. A is the pole of π and P is the pole of α for two O^2, hence for a pencil. This contains a cone K^2 with vertex A and AP is the polar line λ of α relative to K^2; P lies, therefore, on Λ^2. The pencil also contains a cone L^2 with vertex P and π is the polar plane of $\lambda = AP$ relative to L^2. The polar planes of λ relative to the cones L^2 with vertex P form a pencil of planes through a straight line r_p through P. Hence π passes through r_p. If inversely P is an arbitrary point of Λ^2 and π is an arbitrary plane through r_p, A is the pole of π and P the pole of α, relative to a cone L^2 with vertex P as well as relative to a cone K^2 with vertex A; consequently (P, π) has for image any individual of the pencil k^2 to which the intersections of K^2 and L^2 with α belong.

A plane pencil (A, π) with $\pi \neq \alpha$ has as images the ∞^1 pairs of lines k, k' of the involution in A. For A must be the pole of π and of α; consequently O^2 is a cone K^2 with vertex A. The plane pencil (A, α) has all the ∞^2 pairs of lines through A as images.

Summarizing: *the singular plane pencils all have their carriers P on Λ^2. The singular plane pencils that have an arbitrary point P of Λ^2 as carrier, form a pencil of planes. Each of these pencils has as images the k^2 of a pencil to which a pair k, k' of the involution in A belongs. A plane pencil (A, π) has all these pairs as images, the plane pencil (A, α) has the ∞^2 pairs of lines through A as images.*

If P is a point of intersection of a line $A_i\ A_j$ with Λ^2, the cones L^2 with vertex P form a net, hence all the plane pencils with carrier P are singular; their images form the intersections with α of pencils containing the K^2 corresponding to $\lambda = AP$ and the L^2 of the net. Accordingly: *there are* 12 *sheaves of singular pencils.*

The pencil K^2 contains 3 pairs of planes of which the double lines d_1, d_2, d_3, lie on Λ^2. If P is a point on d_i, the polar plane π of A relative to all L^2 is a fixed plane π. Hence the pencil $(P\pi)$ has as image all the k^2 of the net to which belong the pencil which is the intersection of the pencil L^2 with α, and the pair of lines in which the pair of planes that has d_i as double line, cuts α. 6 of the said 12 vertices of singular pencils lie in pairs on the lines d_i.

5. Let π be an arbitrary plane. The O^2 corresponding to the ∞^2 pencils in π must satisfy the condition that A is a pole of π. Hence the ∞^2 pencils in a plane π are represented as the k^2 of a net $\nu(\pi)$. As π cuts all the generatrices of Λ^2, it ensues from § 3 that $\nu(\pi)$ contains all the intersections of α with the cones K^2 with vertices A. These intersections form a pencil ϱ_0.

The ∞^3 *fields of pencils, each consisting of the* ∞^2 *pencils of a plane, are represented in the* ∞^3 *nets through the fixed pencil* ϱ_0.

§ 6. Let P be an arbitrary point. The O^2 corresponding to the ∞^2 plane pencils with vertex P, satisfy the condition that P is the pole of α. *Hence the* ∞^2 *pencils that have P as vertex, are represented as the* k^2 *of a net* $\nu(P)$.

7. The ∞^2 conics k^2 of α form a plane space Σ_5. Any point of Σ_5 gives a k^2, hence also an O^2 through A_1, A_2, A_3 and A_4. In this way the ∞^5 pencils of R_3 are the images of the points of Σ_5. The ∞^2 plane pencils of a plane π of R_3 have as images the points of a plane $\nu(\pi)$ of Σ_5 passing through a fixed straight line ϱ_0. The ∞^2 plane pencils with a point P as vertex are represented on Σ_5 as the points of a plane $\nu(P)$.

The net O^2 consisting of a plane $\alpha_i = A_j\ A_k\ A_l$ and an arbitrary plane through A_i is represented as a plane ν_i of Σ_5. Each pair of planes ν_i, ν_j has a point G_{ij} in common, the image of the pair of planes α_i, α_j. The lines $G_{ij}\ G_{ik}$, $G_{ik}\ G_{il}$, $G_{il}\ G_{ij}$ in ν_i are the images of three pencils O^2 the individuals of which consist of α_i and the planes through $A_i\ A_l$, resp. $A_i\ A_j$, resp. $A_i\ A_k$. Accordingly we find in Σ_5 a configuration of 4 planes ν_i, 12 straight lines $G_{ij}\ G_{ik}$ and 6 points G_{ij}; any plane ν_i contains 3 points G_{ij}, G_{ik}, G_{il}; through each point G_{ij} there pass 4 straight lines to G_{ik}, G_{il}, G_{jk}, G_{jl}; on each straight line there lie 2 points G_{ij}, G_{ik}; through each point G_{ij} there pass 2 planes ν_i and ν_j.

The O^2 consisting of a plane through $A_i\ A_j$ and a plane through

$A_k A_l$ are represented as the points of a quadratic surface $\Omega_{ij} = \Omega_{kl}$ which passes through the 4 lines $G_{ik} G_{il}$, $G_{jk} G_{jl}$, $G_{li} G_{lj}$, $G_{ki} G_{kj}$, which form a skew quadrilateral. Each pair Ω_{ij}, Ω_{ik} has 2 points G_{il} and G_{jk} in common. The 3 pairs of planes mentioned in § 4 of the pencil of cones K^2 with vertex A have as images 3 points $H_{ij} = H_{kl}$ of Σ_5, which lie resp. on the 3 quadratic surfaces Ω_{ij}. The 3 points H_{ij} lie on the line ϱ_0. In this way we have determined the Ω_{ij}, as they are defined by the points H_{ij} and the said skew quadrilaterals.

8. Now we can find something more about the planes $\nu(P)$ of Σ_5 mentioned in § 6. Let P be a point of R_3. The cones L^2 with vertex P form a pencil which is represented as a straight line $\varrho(P)$ of Σ_5. Among them there is a cone L^2 which is formed by the planes $PA_i A_j$ and $PA_k A_l$; accordingly this L^2 is represented as a point of Ω_{ij}. As the plane pencil associated to this L^2 has P as carrier, we find:

Any plane $\nu(P)$ *of* Σ_5, *image of the sheaf of plane pencils* (P), *cuts any* Ω_{ij} *in one point. These 3 points of intersection are collinear.*

9. Now we shall try to find the images of the ∞^4 pencils (P, π) of which π passes through a given point P_0. The associated O^2 must satisfy the necessary and sufficient condition that A and P_0 are conjugated, hence:

The ∞^4 *plane pencils of which the planes pass through a given point* P_0, *have as images in* Σ_5 *the points of a plane four-dimensional space* $V(P_0)$ *through the straight line* ϱ_0. That $V(P_0)$ passes through ϱ_0 results from the fact that any K^2 of the pencil of cones with vertex A produces all the plane pencils of wich the vertices lie on the associated λ, hence even an infinite number of pencils with planes through P_0.

In the same way we have:

the ∞^3 *pencils of which the planes pass through a given line* l, *have as images in* Σ_5 *the points of a plane three-dimensional space* $D(l)$ *through the line* ϱ_0.

In this way the ∞^4 straight lines of R_3 are represented in the plane three-dimensional spaces through a fixed straight line of Σ_5, hence also on a plane four-dimensional space.

10. Let us consider all plane pencils (P, π) for which P is fixed and π turns round a straight line l through P. O^2 must satisfy the necessary and sufficient condition that P is the pole of α and that the polar plane of A passes through l; hence:

The ∞^1 *pencils with fixed vertex* P *of which* π *contains a given straight line* l *through* P *(a pencil of pencils* $[P, l]$*), are represented in* Σ_5 *as the points of a straight line* ϱ (P, l) *lying in the plane* ν (P).

Hence the ∞^5 pencils of pencils are represented as the ∞^5 straight lines of the ∞^3 planes $\nu(P)$.

11. To an arbitrary straight line ϱ of Σ_5 there correspond a pencil of conics in α and a pencil of O^2 in R_3. The polar planes π of A relative to this O^2 pass through a fixed straight line l, and the pole of α describes a cubic γ which cuts l twice. Hence: *to an arbitrary straight line of Σ_5 there corresponds a set of plane pencils (P, π) in which P describes a cubic γ and π passes through a fixed chord of γ.*

12. We try to find the surface $\Omega(l)$ in Σ_5 which represents the ∞^2 plane pencils (P, π) where P describes a line l and π turns round l. $\Omega(l)$ lies in $D(l)$. The image of a straight line of $D(l)$ in R_3 is a pencil O^2, of which all the individuals have the property that the polar plane of A passes through l. The locus of the poles of α is a cubic which cuts l twice; *hence $\Omega(l)$ is a quadratic surface.* The straight line l cuts A^2 twice; hence $\Omega(l)$ cuts ϱ_0 in two points. Let the plane α_i cut l in L_i. There exists a plane through $A_i L_i$ which, together with α_i, forms an O^2 for which the plane (A_i, l) has the point A as pole. Consequently $\Omega(l)$ cuts each of the 4 planes ν_i in one point. The space $D(l)$ cuts Ω_{ij}, besides in H_{ij}, in one more point. This point is the image of a pair of planes O^2 of which the line of intersection s rests on l and for which the plane (s, l) is the polar plane of A; accordingly the pencil (s, l) is associated to it; we see that $\Omega(l)$ cuts the surfaces Ω_{ij}. Hence:

The quadratic surfaces $\Omega(l)$ lie in the three-dimensional spaces $D(l)$ through ϱ_0. $\Omega(l)$ cuts each of the 4 planes ν_i in their points of intersection with $D(l)$ and each of the 3 Ω_{ij} in the points where they cut $D(l)$ outside ϱ_0 [1].

13. Let us now consider the ∞^3 cones with vertices in α and their images in Σ_5. The vertices of the cones of a net O^2 form a curve of the order 6 so that the net contains 6 cones with vertices in α. *Accordingly the ∞^3 cones with vertices in α are represented as the points of a V_3^6 in Σ_5.* This V_3^6 passes through ϱ_0, through the 3 quadratic surfaces Ω_{ij} and the 4 planes v_i. V_3^6 contains the images of the pencils of cones with vertices in α and consists, therefore, of ∞^2 straight lines. By the aid of this V_3^6 we can find more about the ∞^3 planes $\nu(P)$ of Σ_5 and about the $\Omega(l)$. Consider an arbitrary pencil O^2 in the net for which each individual has P as pole of α. This pencil contains 3 cones with vertices in α. These cones are represented as 3 collinear points of V_3^6. Let, inversely, t be a trisecant of V_3^6. It represents a pencil O^2 which contains 3 cones with vertices in α. If P is the vertex of the 4th cone of the pencil, P is the pole of α for all the individuals of the pencil; hence t lies in $\nu(P)$. Consequently (cf. § 10):

The straight lines ϱ (P, l) of Σ_5, images of the ∞^5 pencils of pencils $[P, l]$ of R_3, are the ∞^5 trisecants of V_3^6.

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[1] It is obvious that for "Vereinigte Lage" of two plane pencils it is necessary and sufficient that their image points in Σ_5 lie on the same $\Omega(l)$.

The planes $\nu(P)$ *of* Σ_5, *images of the* ∞^3 *sheaves of pencils of* R_3, *are the* ∞^3 *planes which are formed by those trisecants.*

14. Now we shall explain the two systems of straight lines on each $\Omega(l)$. One system consists of the images of the pencils of pencils $[P, l]$ if P describes the line l. Accordingly $\Omega(l)$ consists of ∞^1 trisecants of V_3^6. In order to explain the other system, we consider an arbitrary straight line l' through A in α. The O^2 for which l and l' are associated, form a pencil. And as the plane pencils that correspond to all the individuals, contain l, the pencil is represented as a straight line of $\Omega(l)$. If l' turns round A in α, there arise ∞^1 pencils of which the images in Σ_5 are the straight lines of the second system on $\Omega(l)$. In particular we find:

All the $\Omega(l)$ *consist of* ∞^1 *trisecants of* V_3^6.

We may also say:

The surfaces $\Omega(l)$ *consist of the trisecants of* V_3^6 *that lie in the 3-dimensional spaces* $D(l)$ *through* ϱ_0.

Let us mention one more property: the $\Omega(l)$ corresponding to the rays l of the pencil (A, α), are split up into 2 planes of which one is fixed, to wit the plane of Σ_5 which represents the net O^2 touching α in A.

Mathematics. — *"A Representation of the Linear Complex of Rays."* By Prof. JAN DE VRIES.

(Communicated at the meeting of March 28, 1925).

§ 1. In order to arrive at a simple representation of the linear complex of rays **L**, we choose two lines a and b that are no complex rays, as double lines of a skew involution. Further we choose a point C so that the transversal t_c through C cutting a and b, does not belong to **L**.

To a complex ray r there corresponds a line l in the involution in question; the transversal t through C cutting l and r, defines on r a point S which we consider as the *image* of r.

If S is an arbitrary point, σ its null plane and (L, λ) the plane pencil which is homologous with the plane pencil (S, σ), S is the image of the ray in σ that corresponds to the ray l which rests on CS.

§ 2. *Singular rays. a.* If l passes through C, S is an arbitrary point of r; this ray belongs to the plane pencil that has the point C^* associated to C, as vertex. Accordingly all the rays r^* in the null plane γ^* of C^* are *singular* and the plane pencil (C^*, γ^*) is its own image.

b. If r lies in the plane (Ca), and accordingly cuts l, S is an arbitrary point of r. Hence the rays r_a in $\varrho_a = (Ca)$ and r_b in $\varrho_b = (Cb)$ are *singular* and the plane pencils (R_a, ϱ_a) and (R_b, ϱ_b) are their own images.

c. The complex rays r_0 resting on a and b, are *singular*, for l_0 coincides with r_0, so that S becomes indefinite. The singular rays r_0 form a *quadratic scroll* $(r_0)^2$.

§ 3. *Image of a plane pencil.* If the plane pencil (L, λ) is associated to the plane pencil (R, ϱ), the transversals t form a quadratic cone. Hence the image of (R, ϱ) is a *conic* ϱ^2. The points of intersection A and B of ϱ with a and b lie on ϱ^2.

The conic ϱ^2 degenerates into a *pair of lines* whenever the plane pencil contains a singular ray, hence a ray r^*, r_a, r_b or r_0; in this case the images of the other rays lie on a straight line u.

The plane pencil can also contain *two singular rays*; in this case the vertex R is the image of all the other rays, hence a *singular point*.

§ 4. *Singular points. a.* If R lies on t_c, t_c contains also the homologous point L; the image of (R, ϱ) consists of two rays r_a and r_b.

Hence any point of t_c is *singular*. The points C and $C^\star$ belong to this group.

b. If A is a point of a, the plane pencil (A, α) contains a ray r_0 and a ray r_a. Accordingly all points of a and b are *singular*.

c. If the plane ϱ_0 of a pencil passes through $C^\star$ and through a ray r_0, the image consists of r_0 and a ray $r^\star$ and the vertex R_0 is a *singular* point. Any ray $r^\star$ cuts two rays r_0; the locus of the *singular points* R_0 is, therefore, a *conic* σ^2 in $\gamma^\star$.

The plane ϱ_0 envelops a quadratic cone. Each of the two planes ϱ_0 through an arbitrary point P contains a plane pencil with vertex R_0. The complex rays that rest on σ^2, form a congruence [2,2].

In the plane (Ca) lies t_c, hence $C^\star$; accordingly it contains a ray $r^\star$ and also a ray r_0, hence a point $R_0 \equiv R_a$. Consequently the vertex of the plane pencil (R_a, ϱ_a) lies on σ^2; the line $C^\star R_a$ cuts σ^2 besides in the intersection of a with the plane $\gamma^\star$.

§ 5. The conics ϱ^2 rest on a, b, t_c and twice on σ^2. The line u_0 which forms with a singular ray r_0 the image of a plane pencil, rests on t_c and on σ^2, for r_0 cuts a, b and σ^2. Consequently the lines u_0 form a *congruence* [2,2] with directrices t_c and σ^2.

The lines $u^\star$, which complete a ray $r^\star$ to a ϱ^2, rest on a and b and form, accordingly, a *congruence* [1,1].

The lines u_a rest on b and on σ^2, for a ray r_a cuts a, t_c and σ^2 (in R_a); hence the rays u_a form a *congruence* [1,2] with directrices b and σ^2.

§ 6. A *field of points* $[S]$ is the *image* of a *congruence* [2,2[. For the two points of intersection of a ϱ^2 with the plane Σ of $[S]$ are the images of two rays r in the plane ϱ. This [2,2] contains *sixteen plane pencils*. In the first place the complex plane pencil of Σ of which the image lies in Σ. Σ contains further two lines u_a, two lines u_b and one line $u^\star$; the *seven* pairs of lines to which they belong, are the images of as many plane pencils of [2,2]. Σ contains *five* singular points; consequently the congruence [2,2] contains a plane pencil (A,α), a plane pencil (B,β), a plane pencil with vertex on t_c and two plane pencils (R_0, ϱ_0). Finally the point ranges on the intersections of $\gamma^\star$, ϱ_a and ϱ_b are the images of the plane pencils $(C^\star, \gamma^\star)$, (R_a, ϱ_a) and (R_b, ϱ_b).

§ 7. *Image of a net of rays.* The rays of **L** resting on a line d, also cut the polar line d' of d. Any plane ϱ through d contains a plane pencil, hence a conic ϱ^2 that cuts d'.

Accordingly the *image* of a bilinear congruence is a *cubic* surface Σ^3 through the lines d and d'; it contains, besides, the lines a, b, t_c and the conic σ^2.

A *parabolic net of rays* with directrix r is represented by a cubic *monoid* that has the image S of r as double point.

§ 8. Two surfaces Σ^3 have a *rational twisted curve* ϱ^4 in common besides a, b, t_c and σ^2; it is the *image* of a *quadratic scroll* $(r)^2$. It has the lines a, b and t_c as *bisecants,* for on each of these lines there rest two rays of $(r)^2$. It has four points in common with σ^2.

§ 9. A *point range* (S) is the image of a *cubic* scroll $(r)^3$, for the locus g of (S) meets three lines r that have their images on the surface Σ^3 corresponding to a [1,1] and, accordingly, cut the directrix d lying on that surface.

As g is the single directrix of $(r)^3$, the polar line g' of g must be the double directrix. $(r)^3$ contains the conic σ^2, for any plane pencil with vertex R_0 contains a ray resting on g. For the same reason a, b and t_c lie on $(r)^3$.

Mathematics. — *"The Triple Involution of Reye"*. By Prof. JAN DE VRIES.

(Communicated at the meeting of April 25, 1925).

1. The congruence of the twisted cubics through five points B_k defines in an arbitrary plane V the triples of an involution that has been examined by REYE [1]). The tangent c at B_5 to the curve ϱ^3 which cuts V in a triple C_1, C_2, C_3, meets V in the point C, which we consider as the *image* of the group (C). For this representation the intersections S_{k5} of the lines $B_5\,B_k$ and V are *singular*; for $B_5\,B_k$ forms a degenerate ϱ^3 with any conic ϱ^2 of the pencil in the plane $B_h\ B_l\ B_m$ that has the points B_h, B_l, B_m and the passage of $B_5\,B_n$ as base points. Accordingly the point S_{k5} is the image of all the triples formed by S_{k5} and the pairs of the involution which are the intersections of these conics and the line $b_{hlm} \equiv S_{hl}\ S_{hm}\ S_{lm}$.

2. If C describes the straight line l, the curves ϱ^3 that touch the plane $B_5 l$ at B_5, describe a surface Λ that has only components of degenerate figures in common with the plane $B_1 B_2 B_3$. Through B_3, B_4, B_5 there passes a ϱ^2 that touches the plane $B_5 l$ at B_5 and cuts $B_1\,B_2$. In accordance with this the intersection of Λ and the plane $B_1 B_2 B_3$ consists of the straight lines $B_1 B_2, B_1 B_3$, and $B_2 B_3$, and Λ is a quadrinodal cubic surface through the edges of the tetrahedron $B_1 B_2 B_3 B_4$ and the point B_5. Accordingly, as Λ is defined by two points C, hence by two triples (C):

"Any two triples of the involution of triples define with the passages D_{kl} of the edges of $B_1 B_2 B_3 B_4$ a cubic λ^3 that contains an infinite number of other triples".

The curves λ^3 form a net that has the angular points D_{kl} of a complete quadrilateral as base points. To this net belongs the configuration consisting of the line $D_{12}\,D_{34}$ and a definite conic λ^2. The corresponding surface Λ consists of the curves λ^3 that rest on $D_{12}\,D_{34}$ [2]).

3. If we define Λ^3 by a point on the line b_{123} and another point chosen at random in V, Λ^3 consists of the plane $B_1 B_2 B_3$ and a quadratic cone with vertex B_4. For the straight line l passes through the point S_{45}, so that Λ^3 contains all the conics ϱ^2 in $B_1\ B_2\ B_3$. Disregarding the groups

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[1]) Geometrie der Lage, 3e Auflage, p. 225.

General considerations of triple-involutions in the plane are found in my paper on "cubic involutions". (These Proceedings **16**, p. 974—987).

[2]) The curves ϱ^3 which cut an arbitrary straight line, form a surface of the 5th order with triple points in B_k. Here it is replaced by Λ^3 and the planes $B_1\,B_2\,B_5$, $B_3\,B_4\,B_5$.

(*C*) consisting of S_{45} and the pairs of the involution that lies on b_{123}, the point range on l is the image of an infinite number of triples lying on a ϱ^2.

"Any conic through the points D_{14}, D_{24}, D_{34}, S_{45} contains an infinite number of groups of the involution of triples."

As any of the five points B can be the vertex of the sheaf that produces the representation, there are *ten* pencils (λ^2) ¹).

4. If the image point C describes a conic γ^2, the corresponding curves ϱ^3 describe a surface Γ^6 with quadruple points B_1, B_2, B_3, B_4, double point B_5 and double lines in the edges of $B_1 B_2 B_3 B_4$. For the plane $B_3 B_4 B_5$ contains two conics ϱ^2 which touch a generatrix of the cone that projects λ^2 out of B_5; hence $B_1 B_2$ is a double line of Γ etc.

The intersection of Γ and V is a curve γ^6 with 6 double points D_{kl}, which contains an infinite number of triples (*C*).

If γ^2 passes through S_{45}, Γ contains the figures of which $B_4 B_5$ is a component part and we find a curve γ^5 with three double points, which contains an infinite number of triples. Etc.

Finally any γ^2 through the four singular points S_{k5} contains an infinite number of triples each of which has a point of γ^2 as image.

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¹) Cf. W. VAN DER WOUDE, *The Cubic Involution of the First Rank in the Plane.* (These Proceedings **12**, p. 751).

Chemistry. — *"Equilibria in the system Zinc sulphate-Water"*. By Prof. ERNST COHEN and Dr. A. L. TH. MOESVELD.

(Communicated at the meeting of February 28, 1925).

1. In connection with the theory of the so-called "Transition elements" ERNST COHEN, some twenty five years ago, carried out a number of solubility determinations for the system zinc sulphate-water [1], at temperatures between — 5° C. and the transition temperature.

These determinations, mentioned in Table 1, were in good agreement with the experiments carried out by CALLENDAR and BARNES at about the same time [2]. The latter also embrace the temperature interval, lying between the transition temperature (39°) and 50° C.

TABLE I.

	Solubility of $ZnSO_4 . 7 H_2O$.				Solubility of $ZnSO_4 . 6 H_2O$.			
Temp.	1st Determination.	2nd Determination.	mean COHEN.	CALLENDAR and BARNES.	1st Determination.	2nd Determination.	mean COHEN.	CALLENDAR and BARNES.
— 5°	39.33	39.27	39.30	—	47.08	—	47.08	—
0°.1	41.94	41.92	41.93	41.85	49.53	49.44	49.48	—
9°.1	47.11	47.07	47.09	46.96	—	54.20	54.20	—
15°.0	50.83	50.94	50.88	50.74	57.09	57.20	57.15	—
25°.0	57.94	57.87	57.90	57.95	63.74	—	63.74	63.74
30°.0	—	—	—	61.92	65.80	65.84	65.82	65.65
35°.0	66.59	66.63	66.61	66.61	67.99	—	67.99	67.94
39°.0	70.00	70.09	70.05	70.05	—	70.08	70.08	70.02

COHEN at the same time, pointed out that the solubility curve of the heptahydrate of zinc sulphate between — 5° and 39° C. may be represented by the equation

$$L_1 = 41.80 + 0.522t + 0.00496t^2 \quad . \; . \; . \; . \; . \; . \; . \quad (1)$$

while the observations of CALLENDAR and BARNES for the hexahydrate between 39° and 50° C. may be represented by the equation:

$$L_2 = 59.34 + 0.0054t + 0.00695t^2 \quad . \; . \; . \; . \; . \; . \; . \quad (2)$$

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[1] Zeitschr. f. physik. Chemie **34**, 179 (1900).

[2] Proceedings Royal Society London **62**, 117 (1897).

The values in the last column of Table 1 were calculated by means of this equation. In either equation L represents the number of grammes $ZnSO_4$, present at $t°$ in 100 gr. of water. Moreover, COHEN pointed out, that the curve (2) gives a good agreement as regards the solubility of the hexahydrate up to 25° C.

In an investigation, lately published by CHARLES R. BURY [1]), the study of the equilibrium in the system zinc sulphate-water was again taken up. His results, given in fig. 1, may be summarized as follows:

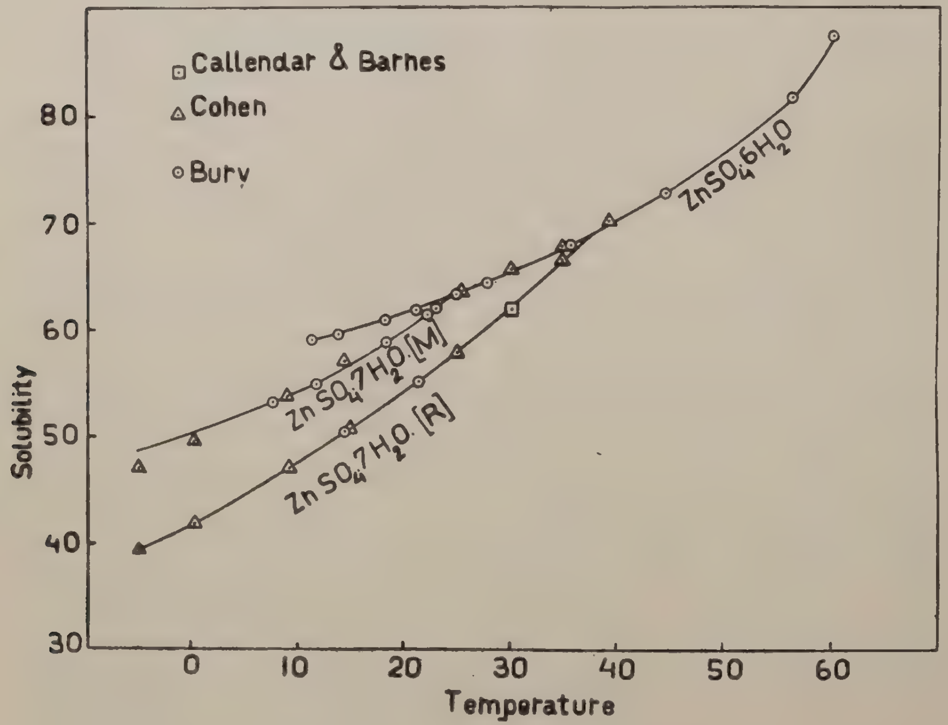

Fig 1.

Between 40° and 50° C. the agreement with the data of CALLENDAR and BARNES was satisfactory. COHEN's figures between 39° and 25° C. (metastable interval of the hexahydrate) agree with BURY's. Below 25° C. the values of the solubility of the hexahydrate found by COHEN lie on another curve than those found by BURY for the hexahydrate. They evidently refer to another hydrate.

Below 11° C. the hexahydrate changes spontaneously and irreversibly into the monoclinic heptahydrate. The latter is most easily obtained by spontaneous crystallisation of supersaturated solutions at about — 10° C.

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[1]) Journ. chem. Soc. London **125**, 2538 (1924).

The values found by BURY *for the solubility of this monoclinic heptahydrate are, between 7° and 23° C., in very close agreement with those given by* COHEN [1]) *for the hexahydrate in this interval.* The monoclinic heptahydrate is metastable at all temperatures with respect to the rhombic heptahydrate. According to BURY the (metastable) monoclinic heptahydrate passes into the metastable hexahydrate at 24°.8 ± 0.3.

3. We wish to observe here that already SCHROEDER [2]), and later on LECOQ DE BOISBAUDRAN [3]) and also STORTENBEKER [4]) have described the monoclinic heptahydrate.

4. In the first place we shall now, by means of another procedure than BURY used, show that his statement, mentioned in 2 (in italics) is perfectly accurate and that therefore COHEN has not determined the solubility of the hexahydrate along the whole range of temperature — 5° up to 39° and 50° C., as he thought, but that in a part of this interval (— 5° up to 23° C.) another hydrate, probably the metastable monoclinic heptahydrate played the part of solute.

5. This proof, which, as will be shown, can be given very clearly, is drawn from a paper by COHEN: "Eine neue Methode zur Bestimmung von Umwandlungstemperaturen" [5]).

In an apparatus, especially made for that purpose, he determined between — 5° and + 50° C. by means of a dipping electrode (F. KOHLRAUSCH) the electrical resistance of saturated solutions of zinc sulphate.

In Table 2 as well as in Fig. 2, taken from COHEN's paper, the results of these measurements are given. In Fig. 2 the curves *CB* and *BA'* relate to the hexahydrate, *BA* to the heptahydrate.

The value of the resistance measured (point X) which, at 25° C., according to Table 2 would relate to the (at that temperature) metastable hexahydrate, falls far outside the curve *BA'*, and as this divergence lies far outside the experimental errors, the question arises, whether the points, furnished by the experiments between *A'* and *C*, really belong to one single continuous curve, or whether we have here two curves, to one of which belongs point X.

6. In order to investigate this we proceeded as follows: it appears to be possible to find a curve representing in the temperature interval of 25°—50° C. the resistances measured as function of the temperature in

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[1]) The italics are C. and M's.

[2]) Lieb. Ann. **109**, 35 (1859).

[3]) Ann. de chim. et de phys. (4) **18**, 266 (1869).

[4]) Zeitschr. f. physik. Chemie **22**, 60 (1897).

[5]) Zeitschr. f. physik. Chemie **31**, 164 (1899).

TABLE II.

Temperature.	Resistance of the solution.	
	Saturated with respect to $Zn\,SO_4 . 7\,H_2O$.	Saturated with respect to $Zn\,SO_4 . 6\,H_2O$.
— 5°.0	445.9	524.1
0.0	384.2	452.2
5.0	337.0	396.3
9.0	305.7	360.3
15.0	271.6	315.5
25.0	236.4	274.2
30.0	225.1	248.8
35.0	218.5	228.3
39.0	215.0	215.0
45.0	—	200.1
50.0	—	191.1

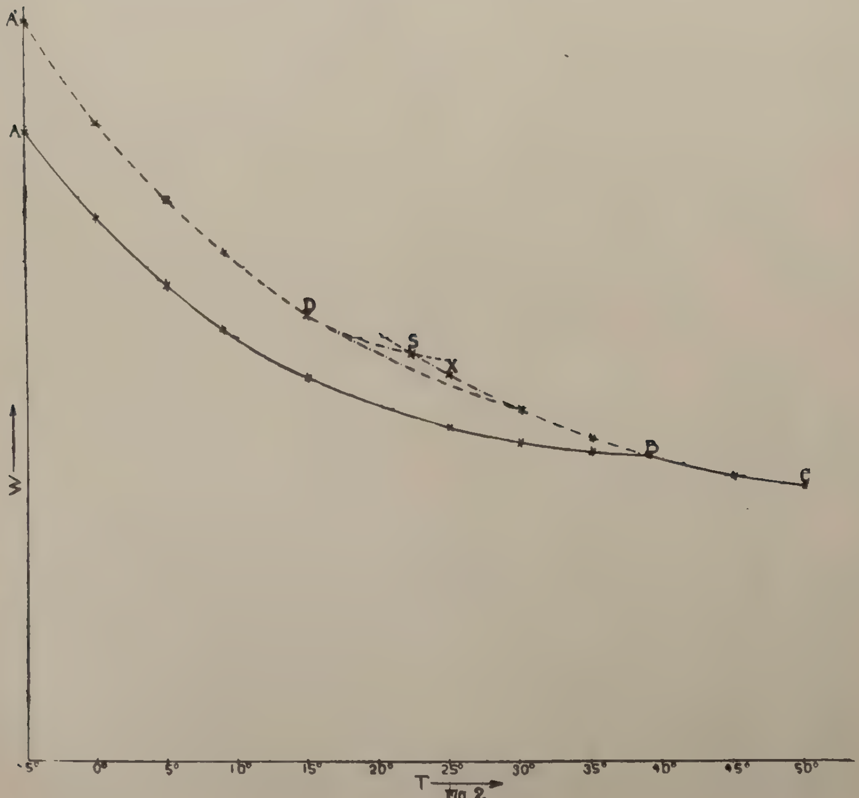

Fig 2

solutions in which the presence of the hexahydrate as a solute was supposed. From the values of Table 2, first and third columns we then find by means of the method of least squares the equation:

$$W_2 = 460.0 - 9.51t + 0.0828t^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

From Table 3, fourth column, we see that this equation describes exactly the observations in this interval.

TABLE III.

t.	$W_{2_{found}}$	$W_{2_{calc.}}$	$W_{2_{found}} - W_{2_{calc.}}$
25°.0	274.2	274.0	— 0.2
30.0	248.8	249.1	+ 0.3
35.0	228.3	228.5	+ 0.2
39.0	215.0	215.0	± 0.0
45.0	200.1	199.7	— 0.4
50.0	191.1	191.4	+ 0.3

As the resistance measurements were made according to KOHLRAUSCH with a bridge wire of 1000 mm., a difference of 1 mm. in reading near the middle of the bridge corresponds with a difference of resistance of 0.4 %. As is seen from column 4 of Table 3 this divergence is not reached anywhere.

If, on the contrary, using the values of resistance measured (Table 2, column 3) we try to calculate a curve for the temperature interval —5° to 25° C., in which the point X is inserted, divergences appear between the values, given by the curve and the values, which have been experimentally determined, which lie far outside the errors of measurement.

If point X is not taken into consideration and with the method of least squares a curve is calculated by means of the figures of Table 2, column 3, between —5° and 15° C. we find for that curve the equation:

$$W_1 = 453.8 - 12.70t + 0.2354t^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

which, as is shown by Table 4 describes the observations most satisfactorily.

TABLE IV.

t.	$W_{1_{found}}$	$W_{1_{calc.}}$	$W_{1_{calc.}} - W_{1_{found}}$
— 5°.0	524.1	523.2	— 0.9
0.0	452.2	453.8	+ 1.6
5.0	396.3	396.2	— 0.1
9.0	360.3	358.6	— 1.7
15.0	315.5	316.2	+ 0.7

If, by means of (4) the value of the resistance at 25° C. is calculated, we find a value, which considerably differs from the one found experimentally (calc. 282.2; found 274.2).

7. The curves (3) and (4) therefore refer to the saturated solutions of two different substances (solutes). If the temperature of the intersection *S* of curves (3) and (4) is calculated we find for it 22°.5 C.

If we now take into consideration that BURY's values for the solubility of $ZnSO_4 . 7$ aq. monoclinic between 7° and 23° agree with those given by COHEN for $ZnSO_4 . 6$ aq in this temperature interval, and that BURY has found 24°.8 ± 0.3 for the temperature of the intersection of the curves, which refer to $ZnSO_4 . 7$ aq monoclinic and $ZnSO_4 . 6$ aq, whereas our curves intersect at 22°.5 C. (curves (3) and (4)), it is highly probable, that curve (4) refers to $ZnSO_4 . 7$ aq monoclinic, and is not, as COHEN thought the production of (3) and therefore does not refer to $ZnSO_4 . 6$ aq.

If COHEN had examined in his apparatus the solute which was present at the determinations between —5° and + 25° he would undoubtedly have perceived that he had to do with the monoclinic (metastable) hydrate of $ZnSO_4 . 7$ aq.

8. BURY's researches on the one side furnish us with an explanation of the fact that the point *X*, found by COHEN in his resistance determinations falls totally outside the curve of the resistances, while on the other side the present point of view shows, that these resistance and solubility determinations by COHEN are in good agreement with BURY's results, if we apply the figures of the solubility and resistance found by COHEN below 25° C. to the (metastable) monoclinic hydrate ZnSO. 7 aq instead of to the (metastable) ZnSO. 6 aq.

9. The above demonstrates how important it is to mention "discrepant" results, when giving results of an investigation. In the case which we have discussed here, it is the mentioning of point *X*, (Fig. 2) which has facilitated the checking of the results, which have been obtained in a different way.

SUMMARY.

It has been demonstrated that BURY's investigations on the equilibria in the system zinc sulphate-water, made by means of solubility determinations, give a complete explanation of the results, which at the time COHEN obtained by the same method as well as by means of determinations of resistance.

Utrecht, February 1925. VAN 'T HOFF-*Laboratory*.

Physics. — *"On the light emitted by gases in the solid state and the spectrum of the northern light"*. By L. VEGARD, H. KAMERLINGH ONNES and W. H. KEESOM. (Communication from the physical laboratory at Leyden)[1].

(Communicated at the meeting of March 28, 1925).

This paper contains the results we obtained in the Leyden laboratory in cooperation with Mr. I. BERGE, and which form a continuation of the researches described by one of us in preceding communications[2]) on the light emitted by gases in the solid state.

After a series of investigations made at the temperature of liquid hydrogen, we observed the luminescence of different gases and gaseous mixtures at the temperature of liquid helium. We shall describe here especially the investigations on the light emitted when solid nitrogen and mixtures of nitrogen and neon at the temperature of *liquid helium* are bombarded by swift cathode rays.

From the light emitted by pure nitrogen at the temperature of liquid helium we obtained several spectra of high dispersion, which show, that the band N_1 doubtlessly has the same structure and place as at the temperature of liquid hydrogen. In this case there is a principal maximum (5555) with two secundary maxima. This result is in correspondance with former observations made with liquid hydrogen under reduced pressure and shows that the *band N_1 does not become narrower at lower temperatures*. The band N_2 is strong, but is split up now into two components (5236, 5222).

Besides the bands N_1 and N_2 a *regular series of bands* is most prominent in this spectrum. Each band consists of a double line and a diffuse edge towards the red. In this respect they are different from the bands emitted by gaseous nitrogen. With the glass spectrograph used we obtained 7 bands on the plate, but doubtlessly these extend into the ultraviolet region.

The wavelengths are the following:

	1°.	2°.	3°.	4°.	5°.	6°.	7°.
strong	5770	5490	5116	4784	4490	4223	3986
weak		5502	5127	4795	4499	4231	3994.

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[1]) A more detailed description will appear in one of the next "Communications" from the physical laboratory at Leyden.

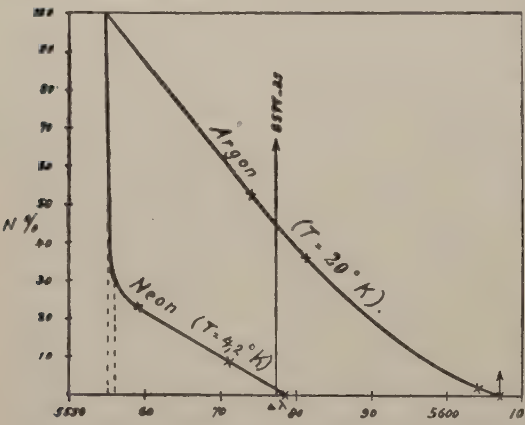
[2]) L. VEGARD. These Proceedings 27, p. 113. Comm. from the phys. lab. at Leyden N⁰. 168*d*, C. R. t. 176 p. 974, t. 176 p. 1488, 1923, t. 178 p. 1153, t. 179 p. 35, t. 179 p. 151, 1924.

Undoubtedly we have here a state of development of the same bands that have already been observed at the temperature of liquid hydrogen and which became very strong with canal rays [1]). At lower temperatures these bands, which at higher temperatures are very broad and diffuse, are contracted to narrow lines with a diffuse edge towards the read.

In argon containing traces of nitrogen and at the temperature of liquid hydrogen the bands are also contracted into two lines, but these lines have another position. The two strongest bands, corresponding to the fifth and the sixth band of the table, have the wavelengths: (4523, 4473) and (4231, 4211).

In a former communication one of us has remarked [2]), that doubtlessly these two bands correspond with the bands observed by Lord RAYLEIGH in the diffuse light of the night sky, but as long as the wavelengths of the bands of Lord RAYLEIGH have not been determined more accurately we cannot say to which form of contraction they belong.

At the same temperature of liquid helium, nitrogen emits also three or four weak lines in the region of the great wavelengths viz. (5914, 5952, 6399) (6417?), which probably correspond to weak lines of the northern light spectrum.



The *experiments with solid neon* containing nitrogen in different proportions have shown that the band N_1 changes in a way similar to that found earlier for mixtures of nitrogen and argon [3]).

A decrease of the concentration of the nitrogen brings the principal maximum to move towards greater wavelenghts, while the secundary maxima finally disappear. This behaviour of nitrogen-neon mixtures is similar to that of nitrogen-argon-mixtures, but as regards the magnitude of the displacement and the law it follows, the two kinds of mixtures show a very interesting and typical difference, which is illustrated by the figure. Here the wavelength of the principal maximum has been plotted as a function of the nitrogen concentration for the two mixtures. The shift of the principal maximum is caused by a specific action of the admixed gas on one side and by a decrease of the dimensions of the nitrogen particles on the other side.

For the argon mixtures at the temperature of liquid hydrogen the two influences are of the same order of magnitude. The specific action of the argon is evident from the fact that small argon concentrations cause a considerable shift of the principal maximum even when they cannot exert perceptible influence on the magnitude of the nitrogen particles.

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[1]) L. VEGARD. C.R. 179 p. 36.

[2]) loc. cit.

[3]) loc. cit.

For the nitrogen-neon-mixtures however the specific influence of the inert gas is very small. For neon concentrations varying from 0 % up to 70 % the position of the principal maximum remains nearly constant.

At this concentration the maximum begins to move quickly on account of the diminution of the nitrogen particles.

In correspondance with the interpretation of the northern light spectrum formerly published by one of us, we see that when the nitrogen concentration approaches zero, the band N_1 *changes into a line with wavelength 5578.6 which nearly coincides with the green line of the northern light. According to this interpretation, this small difference* $\triangle \lambda = 1.2$ Å *should be accounted for by the specific influence of the neon.*

In neon containing traces of nitrogen the band N_2 is split up into three components (5229—5220—5203) the first of which is the stronger one.

At increasing concentration of the nitrogen, the band N_2 changes into a double line. Moreover the spectrum shows a number of lines perhaps belonging to the neon spectrum.

Zoology. — "*Cross-breeding experiments with the three-and tenspined Stickleback* (*Gasterosteus aculeatus* L. *and Gasterosteus pungitius* L.)." By Dr. G. J. van Oordt. (Communicated by Prof. J. Boeke).

(Communicated at the meeting of May 30, 1925).

It is well-known that interspecific hybridization, even with closely related species, is accompanied by great difficulties. It is remarkable that in many fish-species this is not the case, however. Hybrids between different species, genera, families and even orders of fishes can be easily produced artificially. In most cases these hybrids prove to have great vitality; as a rule the more unrelated the parents are the sooner the hybrids die.

In this respect the experiments of several American investigators (e.g. Moenkhaus 1904, Bancroft 1912, Morris 1914, Newman 1914, 1915, 1918, Pinney 1918, 1922) as well as those of Kammerer (1907), G. and P. Hertwig (1914) a.o. are well-known. As appears from the investigations of Gerschler (1914) and Bellamy (1922) some Cyprinodonts easily hybridize under natural conditions. These hybrids (*Platypoecilus* ♂ × *Xiphophorus* ♀ as well as the reciprocal cross) are absolutely fertile, *inter se* as well as with the parents.

Formerly it was tried to fertilize eggs of *Gasterosteus aculeatus* with sperm of other fish-species; but as far as I know, results were only obtained artificially. It was not only possible to cross *G. aculeatus* with the closely related *Spinachia vulgaris* (Appellöf 1894), but also to fertilize *G. aculeatus*-eggs with sperm of different (some far-related) species and genera (Newman 1915). I do not know whether hybrids between the three closely related species *Gasterosteus aculeatus*, *G. pungitius* and *Spinachia vulgaris* have been obtained under natural conditions. Moreover, hybrids between *G. aculeatus* and *G. pungitius*, both common species in fresh water, are never found in nature. This must be ascribed to the markedly different breeding instincts of both species.

The males of these *Gasterosteus*-species make nests, in which the ♀♀ deposit their eggs. The male of *G. aculeatus* prepares a nest of small branches, leaves of waterplants, roots, sand-grains etc. which is situated at the bottom of the ditch or in the sand of the aquarium. The nest of *G. pungitius*, however, is fastened by preference at some distance from the bottom among dense vegetation. It is composed of about the same material (sand-grains excepted) as the nest of the three-spined Stickleback. In both species the nest material is fastened by means of a secretion, formed by the kidney of the male fish, which hardens in water. The entrance of the nest of *G. pungitius* is at the side, that of *G. aculeatus* is situated closer to the upperside.

In both species the male tries to lure the ripe female into the nest or to chase it into it. When this succeeds the ♀ deposits its eggs in the nest and then the ♂ ejaculates its sperm over the ova. The male guards the nest, drives off all intruders fiercely and constantly conducts fresh water to the eggs by regularly and quickly moving its pectoral fins, while remaining at the entrance of the nest.

The nuptial colours of the males of both species differ markedly. That of *G. pungitius* is almost totally black; the ventral side of the ♂ of *Gasterosteus aculeatus* is brilliant red and the upper side a dark colour; moreover the iridescent eye of the ♂ is very conspicuous. The ♀♀ of *G. aculeatus* are generally paler than those of *G. pungitius*, which are markedly dotted; the depth of the body of the former is relatively greater than that of the latter. The number of dorsal spines of *G. aculeatus* is 3 (two large and one small one), that of *G. pungitius* varies from 9—11.

In the beginning of 1925 several aquaria were prepared, in two of which were placed one male of *G. pungitius* with some females of *G. aculeatus*, in two others one ♂ of *G. aculeatus* with some ♀♀ of *G. pungitius*. Moreover, several ♂♂ of *G. aculeatus* were placed in a larger tank with several ♀♀ of *G. pungitius*, while finally one ♂ of *G. pungitius* was put together with some ♀♀ of the same species. It was the intention to replace the latter by ripe ♀♀ of *G. aculeatus*, when the nest was ready. Care was taken of the size of the fishes being almost equal: generally the specimens of *G. aculeatus*, especially the ♀♀, are somewhat larger than those of *G. pungitius*.

In all cases the ♂♂ assumed their nuptial colouring and soon nest-building began. Though the ♂♂ behaved in the same way towards the ♀♀ of the other species as towards the ♀♀ of the own species, no eggs were laid in the nests, one case excepted. This exception will be described elaborately here.

A male of *Gasterosteus pungitius* was placed together with three females of *G. aculeatus* on April 1. On April 3 it began to make a nest among dense waterplants at about 25 cm. from the bottom of the aquarium. That same day two ♀♀, with swollen abdomens, deposited at the bottom of the aquarium an egg-mass, which was soon devoured by the male. On April 5 both these ♀♀ were dead. The male, which had chased the ♀♀ vigorously, continued chasing the remaining one. In the meantime the nest was completed. On April 9 the ♀ had markedly swollen gonads; it was so heavy that it had great difficulty in swimming and constantly sank to the bottom of the tank. The ♂ often concentrated its attention on the ♀, which remained near the bottom of the aquarium;

it tried to lure the ♀ to the nest, which was situated higher up, but the ♀ could not follow the ♂, owing to the heaviness of its body. In the afternoon of April 9 the relation had somewhat changed: I got the impression that now the ♀ tried to lure the ♂ to the bottom of the tank i. e. to the place, where the nest of G. *aculeatus* is normally found. Consequently a long battle arose.

On April 10 the ♀ had deposited its eggs, its abdomen being slender again. However, the eggs were not found at the bottom of the aquarium; so they must have been eaten or laid in the nest. The nest was not investigated in order not to disturb the ♂. On April 12 the ♀ again possessed a swollen abdomen; the male did not take notice of the nest; it came only a few times near it and only now and then conducted a small quantity of fresh water to it. It must be mentioned that before the eggs were in the nest, the ♂ did this continually. On April 14 the ♀ had deposited its eggs again. On investigating the nest, I found that it contained a great number of fertilized eggs with well developed germ-discs. Taking into consideration the developmental stage, I am of opinion that the eggs were laid on April 13.

Soon after the nest was taken away, the ♂ began to gather new material and on April 15 a new nest was ready. The ♂ also began to chase the ♀ more vigorously; hence it appears that the earlier breeding-instincts begin to show again after the nest has been destroyed.

The nest containing the eggs was placed in a tank with many water-plants. For the first days the mortality of the embryos was small, but soon increased and for that reason the living ova were further bred separately, and the nest was removed. On April 18 the optic vesicles were clearly perceptible; on April 20 pigment was visible in the eyes and the pulsating heart was conspicuous. On April 21 numerous melanophores were formed; then the blood-stream in the large yolk-veins was distinctly visible. On April 22 the first body-movements of the embryo, still in the egg-shell, were observed; then the mortality was so heavy that the few remaining living embryos were fixed (April 23).

The ♀ mentioned above once more showed a strongly swollen abdomen, owing to the numerous ripe eggs, but eggs were not laid in the nest anymore. In the other cases, in which a ♂ of G. *pungitius* and ♀♀ of G. *aculeatus* were put together, I had no success. The females of G. *pungitius*, which had been placed with a male of G. *aculeatus* never even laid eggs. The ♂♂ continually chased the females of the other species. The latter were very much afraid and hid as much as possible among the waterplants. The males of G. *aculeatus* always made a normal nest, however, and though these nests did not contain eggs the males regularly conducted fresh water to them with their pectoral fins. The males of both species never ejaculate their sperm spontaneously. In such „overripe" males the abdomen swells, owing to the retention of the sperm.

I have already mentioned that in „overripe" ♀♀ the abdomen was swollen as well. In a few cases I found that the ♀♀ of *G. aculeatus* with swollen abdomens, deposited their eggs at the bottom of the tank. Sometimes these females died soon after, sometimes they died without having deposited their ova. This was also observed in a few "overripe" ♂♂ of *G. aculeatus* and *G. pungitius*. The possession of such overripe ovaria or testes seems to be very harmful for these specimens.

From the above it follows:

I. That the males of both *Gasterosteus aculeatus* and *G. pungitius* behave towards the females of the other species as if the latter belong to their own species. The fact that the females of *G. pungitius* are more afraid of a male of *G. aculeatus* than of a male of their own species is probably due to the male of *G. aculeatus* chasing them with greater vigour. The behaviour of the ♂♂ of *G. aculeatus* shows, however, that they do not consider the ♀♀ of *G. pungitius* to belong to another species.

II. That the males of both species begin to prepare nests in the presence of females of the other species.

III. That the strongly differing breeding-instincts of both species probably cause the non-despositing of the eggs in the nests of the other species. The case, described above, in which this did happen, was probably due to incidental circumstances.

IV. That the ♂ of *G. pungitius* appears to recognize its eggs only as such, when they are in the nest [1]), for eggs found at the bottom of the aquarium are soon devoured by it.

V. That, when a nest with eggs of *G. pungitius* is taken away, the male at once begins to make a new nest and to chase females. When there are eggs in the nest the ♂ does not chase other specimens with as much vigour.

I cannot trace whether the hybrids thus obtained show characteristics of both parents or of one only, because they are too young in many respects. As the melanophores have developed, however, an investigation of the place where they are found and of the shape of these cells may perhaps decide this question. Moreover, the behaviour of the paternal and maternal chromatin will be traced. I think it very improbable that the eggs develop parthenogenetically, taking into consideration the investigations on other Teleost-hybrids.

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[1]) A similar case was described recently by PORTIELJE (1925) in the Nandu (*Rhea americana* L.) The eggs of this bird are incubated by the male only.

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Zoological Laboratory, School of Veterinary Medicine.

Utrecht, May 1925.

Mathematics. — *"On the binomial equation of prime degree"*. By Prof. J. C. KLUYVER.

(Communicated at the meeting of April 25, 1925).

1. The roots of the equation

$$z^p - 1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where p is prime, can be expressed by radicals and the index of the radical signs involved is $p-1$. Likewise, putting $p-1$ equal to $\nu\nu_1$, the equation of degree ν_1, the roots of which are the socalled GAUSSIAN periods of order ν [1]), admits of an algebraic solution, and the radical signs involved have the index ν_1.

These GAUSSIAN periods are defined as follows. Let g be a primitive root of the prime number p, α any complex root of the equation (1), then, putting

$$\alpha_k = \alpha^{g^k} \quad , \quad \alpha_0 = \alpha,$$

the periods of order ν are the ν_1 sums

$$y_s = \sum_{k=0}^{k=\nu-1} \alpha_{k\nu_1+s} \, .$$

$$(s = 0, 1, 2, \ldots \nu_1 - 1).$$

The method of finding the values of these periods is based upon the consideration of the sum

$$f(\alpha, \gamma^\mu) = \sum_{k=0}^{k=p-2} \alpha_k \gamma^{k\mu},$$

where μ is any integer and γ represents a primitive root of the new binomial equation

$$z^{\nu_1} - 1 = 0 \, . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2)$$

Evidently we have

$$\left.\begin{array}{l} f(\alpha, \gamma^{\mu_1}) = f(\alpha, \gamma^{\mu_2}) \ , \quad \text{if} \quad \mu_1 \equiv \mu_2 \, , \\ f(\alpha, \gamma^{\mu}) = \quad -1 \quad , \quad \text{if} \quad \mu \equiv 0, \end{array}\right\} (mod\, \nu_1)$$

$$f(\alpha_h \, , \gamma^\mu) = \gamma^{-h\mu} f(\alpha \, , \gamma^\mu),$$

and we can prove

$$| f(\alpha \, , \gamma^\mu) | = \sqrt{p} \quad , \quad \text{if} \quad \mu \not\equiv 0 \, , \quad (mod\, \nu_1).$$

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[1]) Disquisitiones arithmeticae (deutsch von H. MASER), Art. 343.

It is then readily seen that the expression

$$\frac{f(\alpha\,,\gamma^{\mu_1})\,f(\alpha\,,\gamma^{\mu_2})}{f(\alpha\,,\gamma^{\mu_1+\mu_2})}$$

is a symmetric function of the roots α_k, depending only on μ_1, μ_2 and γ, and that, when we write

$$\frac{f(\alpha\,,\gamma^{\mu_1})\,f(\alpha\,,\gamma^{\mu_2})}{f(\alpha\,,\gamma^{\mu_1+\mu_2})} = \psi_{\mu_1,\mu_2}(\gamma),$$

the righthand side will be in general a polynomial in γ.

Indeed, when we have $\mu_1 \not\equiv 0$, $\mu_2 \not\equiv 0$, $\mu_1 + \mu_2 \not\equiv 0$, $(mod\ \nu_1)$, it is possible to show that

$$\psi_{\mu_1,\mu_2}(\gamma) = \sum_{r_1, r_2} \gamma^{r_1\mu_1 + r_2\mu_2},$$

where r_1 and r_2 take all the values $1, 2, 3, \ldots\ldots, p-2$, always remaining connected by the one to one relation

$$g^{r_1} + g^{r_2} \equiv 1\ ,\ (mod\ p) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (3)$$

Hence, in the general case, $\psi_{\mu_1,\mu_2}(\gamma)$ is a sum of $p-2$ terms, each of them being a certain power of the root γ, and from the definition it follows that at the same time we have

$$|\,\psi_{\mu_1,\mu_2}(\gamma)\,| = \sqrt{p}.$$

However, when μ_1 and μ_2 are not quite arbitrary, other results are obtained, and in particular we must note the equations

$$\left.\begin{array}{r} \psi_{\mu_1,\mu_2}(\gamma) = -1\ ,\ \text{if}\ \ \mu_1 \equiv 0\ ,\ \text{of als}\ \mu_2 \equiv 0 \\ \psi_{\mu_1,\mu_2}(\gamma) = (-1)^{\mu_1\nu-1}\,p = (-1)^{\mu_2\nu-1}\,p,\ \text{if}\ \mu_1 \not\equiv 0,\ \mu_2 \not\equiv 0,\ \mu_1 + \mu_2 \equiv 0 \end{array}\right\}(mod\ \nu_1).$$

Now, supposing one of the two indices to be equal to unity, we have in particular

$$\left.\begin{array}{ll} & \psi_{1,\mu}(\gamma) = G_\mu(\gamma) = \sum\limits_{r_1, r_2} \gamma^{r_1 + \mu r_2},\ \text{if}\ \ \mu \not\equiv 0,\ \mu + 1 \not\equiv 0,\ (mod\ \nu_1) \\ \text{and also} & \psi_{1,\mu}(\gamma^k) = G_\mu(\gamma^k) = \sum\limits_{r_1, r_2} \gamma^{k(r_1 + \mu r_2)},\ \text{if}\ k\mu \not\equiv 0,\ k(\mu+1) \not\equiv 0,\ (mod\ \nu_1) \end{array}\right\} \quad (4)$$

in which formulae r_1 and r_2 are still connected by the relation (3), and the special sum $G_\mu(\gamma)$ will satisfy the relations

$$\left.\begin{array}{rl} G_{\mu_1}(\gamma) = G_{\mu_2}(\gamma) & ,\ \text{if}\ \ \mu_1 \equiv \mu_2, \\ G_\mu(\gamma) = -1 & ,\ \text{if}\ \ \mu \equiv 0, \\ G_\mu(\gamma) = (-1)^{\nu-1}\,p & ,\ \text{if}\ \ \mu + 1 \equiv 0, \\ |G_\mu(\nu)| = \sqrt{p},\ G_\mu(\gamma) = (-1)^\nu\,G_{-\mu-1}(\gamma), & \text{if}\ \ \mu \not\equiv 0,\ \mu + 1 \not\equiv 0, \end{array}\right\}(mod\ \nu_1) \quad (5)$$

$$G_\mu(\gamma^k) = \frac{G_1(\gamma)\,G_2(\nu)\,G_3(\nu)\ldots G_{(\mu+1)k-1}(\gamma)}{G_1(\gamma)\,G_2(\gamma)\ldots G_{k-1}(\gamma)\,.\,G_1(\gamma)\,G_2(\gamma)\ldots G_{\mu k-1}(\gamma)} \quad . \quad . \qquad (6)$$

Finally, it should be noticed that the sums $f(\alpha, \gamma^m)$ can be expressed by means of $G_\mu(\gamma)$ and that we can establish the equations

$$\left.\begin{aligned} f(\alpha, \gamma)^m &= G_1(\gamma)\, G_2(\gamma)\, G_3(\gamma) \ldots G_{m-1}(\gamma)\, f(\alpha, \gamma^m), \\ f(\alpha, \gamma)^{\nu_1} &= G_0(\gamma)\, G_1(\gamma)\, G_2(\gamma) \ldots G_{\nu_1-1}(\gamma). \end{aligned}\right\} \quad . \quad . \quad (7)$$

Hence, as soon as the sums $G_1(\gamma), G_2(\gamma), \ldots G_{\nu_1-1}(\gamma)$ are calculated, we find the value of $f(\alpha, \gamma)$ by extracting a ν_1-th root, and subsequently we are able to determine $f(\alpha, \gamma^m)$ for any value of m.

Now, putting

$$\nu_1\, y + 1 = \sum_{k=1}^{k=\nu_1-1} f(\alpha, \gamma^k),$$

or

$$\nu_1\, y + 1 = f(\alpha, \gamma) + \sum_{k=2}^{k=\nu_1-1} \frac{f(\alpha, \gamma)^k}{G_1(\gamma)\, G_2(\gamma) \ldots G_{k-1}(\gamma)}, \quad . \quad . \quad . \quad (8)$$

and introducing successively into this formula the ν_1 values of $f(\alpha, \gamma)$, we can show that the ν_1 values found for y are precisely the ν_1 GAUSSIAN periods y_s of the order ν.

It should be noted that if we replace g by another primitive root of p, or γ by another primitive root of the equation (2) the formula (8) still gives the same results, only the order in which the ν_1 values of the periods y_s present themselves will be altered.

2. For a given large prime number p the calculation of the sums $G_\mu(\gamma)$ by means of (4) is very laborious and almost impracticable, but when ν_1 is small, it is sometimes possible to form the equation of the ν_1-th degree whose roots are the ν_1 periods. To this end, we must observe that this equation, the leading coefficient being unity, necessarily has real and integer coefficients, and we must have recourse to the theory of numbers. In this note I will show how to deduce the equation of the periods when ν_1 is 5 or 6, and how to solve it algebraically.

In the first place, let $p = 10\,w + 1$ and $\nu_1 = 5$, then we have according to (5), (7) and (8)

$$\gamma^4 + \gamma^3 + \gamma^2 + \gamma + 1 = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (9)$$

$$\left.\begin{aligned} G_0(\gamma) = -1,\ G_2(\gamma) = G_1(\gamma^2),\ G_3(\gamma) = G_1(\gamma),\ G_4(\gamma) = -p, \\ G_1(\gamma^{-1}) = \frac{p}{G_1(\gamma)},\ G_2(\gamma^{-1}) = \frac{p}{G_2(\gamma)}, \end{aligned}\right\}$$

$$f(\alpha, \gamma)^5 = p\, G_1^2(\gamma)\, G_2(\gamma),$$

$$5y + 1 = f(\alpha, \gamma) + \frac{f(\alpha, \gamma)^2}{G_1(\gamma)} + \frac{f(\alpha, \gamma)^3}{G_1(\gamma)\, G_2(\gamma)} + \frac{f(\alpha, \gamma)^4}{G_1^2(\gamma)\, G_2(\gamma)},$$

or

$$5y + 1 = f(\alpha, \gamma) + \frac{p}{f(\alpha, \gamma)} + \frac{f(\alpha, \gamma)^2}{G_1(\gamma)} + \frac{p\, G_1(\gamma)}{f(\alpha, \gamma)^2},$$

and since $|f(\alpha, \gamma)| = |G(\gamma)| = \sqrt{p}$, it appears that everyone of the five roots y is real. The equation of the periods $F(y) = 0$ presents itself in the form

$$0 = \begin{vmatrix} -\frac{5y+1}{\sqrt{p}} & 1 & \frac{G_1(\gamma^{-1})}{\sqrt{p}} & \frac{G_1(\gamma^{-1})\,G_2(\gamma^{-1})}{\sqrt{p^2}} & \frac{G_1^2(\gamma^{-1})\,G_2(\gamma^{-1})}{\sqrt{p^3}} \\ 1 & -\frac{5y+1}{\sqrt{p}} & 1 & \frac{G_1(\gamma^{-1})}{\sqrt{p}} & \frac{G_1(\gamma^{-1})\,G_2(\gamma^{-1})}{\sqrt{p^2}} \\ \frac{G_1(\gamma)}{\sqrt{p}} & 1 & -\frac{5y+1}{\sqrt{p}} & 1 & \frac{G_1(\gamma^{-1})}{\sqrt{p}} \\ \frac{G_1(\gamma)\,G_2(\gamma)}{\sqrt{p^2}} & \frac{G_1(\gamma)}{\sqrt{p}} & 1 & -\frac{5y+1}{\sqrt{p}} & 1 \\ \frac{G_1^2(\gamma)\,G_2(\gamma)}{\sqrt{p^3}} & \frac{G_1(\gamma)\,G_2(\gamma)}{\sqrt{p^2}} & \frac{G_1(\gamma)}{\sqrt{p}} & 1 & -\frac{5y+1}{\sqrt{p}} \end{vmatrix}$$

and expanding the determinant, we get

$$\begin{aligned} 0 = {} & (5y+1)^5 - 10p\,(5y+1)^3 - 5p\,(5y+1)^2\,[G_1(\gamma) + G_1(\gamma^{-1}) + G_2(\gamma) + G_2(\gamma^{-1})] + \\ & + 5p\,(5y+1)\,[p - \{G_1(\gamma) + G_1(\gamma^{-1})\}\,\{G_2(\gamma) + G_2(\gamma^{-1})\}] - \\ & - p\,[G_1^2(\gamma)\,G_2(\gamma) + G_1^2(\gamma^{-1})\,G_2(\gamma^{-1}) + G_2^2(\gamma)\,G_1(\gamma^{-1}) + G_2^2(\gamma^{-1})\,G_1(\gamma)]. \end{aligned}$$

Now we put

$$G_1(\gamma) = \sqrt{p}\,e^{i\theta_1},\quad G_2(\gamma) = \sqrt{p}\,e^{i\theta_2}),\quad S_1 = G_1(\gamma) + G_1(\gamma^{-1}) + G_2(\gamma) + G_2(\gamma^{-1}) = $$
$$= 2\sqrt{p}\,(\cos\theta_1 + \cos\theta_2),$$

$$S_2 = \{G_1(\gamma) + G_1(\gamma^{-1})\}\,\{G_2(\gamma) + G_2(\gamma^{-1})\} = 4p\cos\theta_1\cos\theta_2,$$
$$\eta = 8p\sqrt{p}\,\sin\theta_1\,\sin\theta_2\,(\cos\theta_1 - \cos\theta_2),$$

whence we have the inequalities

$$|S_1| < 4\sqrt{p}\ ,\quad |S_2| < 4p\ ,\quad |\eta| < 16p\sqrt{p}\ .\ .\ .\ \tag{10}$$

Transforming the equation of the periods $F(y) = 0$ and introducing S_1, S_2 and η we get

$$\left.\begin{aligned} 0 = y^5 + y^4 - \frac{2}{5}(p-1)\,y^3 + \frac{1}{5^2}\,y^2\,(2 - 6p - pS_1) + \\ + \frac{1}{5^3}\,y\,(1 - 6p + p^2 - 2p\,S_1 - p\,S_2) + \\ + \frac{1}{5^5}\left(1 - 10p + 5p^2 - 5p\,S_1 + p^2\,S_1 - 5p\,S_2 - \frac{1}{2}\,p\,S_1\,S_2 + \frac{1}{2}\,p\eta\right) \end{aligned}\right\} \tag{11}$$

in which equation, the prime number p being given, the three numbers S_1, S_2 and η must be evaluated. To this end I proceed as follows.

The equation (4)

$$G_1(\gamma) = \sum_{r_1, r_2} \gamma^{r_1 + r_2}$$

can be reduced to

$$G_1(\gamma) = Q_1 \gamma + Q_2 \gamma^2 + Q_3 \gamma^3 + Q_4 \gamma^4,$$

and as we have $G_2(\gamma) = G_1(\gamma^2)$, the sums $G_2(\gamma)$, $G_1(\gamma^{-1})$ and $G_2(\gamma^{-1})$ are expressible by the same coefficients Q_h that necessarily are integers. Addition of $G_1(\gamma)$, $G_1(\gamma^{-1})$, $G_2(\gamma)$ en $G_2(\gamma^{-1})$ gives at once

$$\sum_1^4 Q_h = -S_1, \quad \dots\dots\dots\dots \quad (12)$$

héence S_1 is an integer. Now writing $t_h = \gamma^h + \gamma^{-h}$, and noticing that we have

$$t_1 = t_4, \quad t_2 = t_3, \quad t_1 + t_2 = -1, \quad t_1 t_2 = -1, \quad (t_1 - t_2)^2 = 5,$$

the condition $|G_1(\gamma)| = \sqrt{p}$ gives

$$p = \sum_1^4 Q_h^2 + t_1 (Q_1 Q_2 + Q_2 Q_3 + Q_3 Q_4) + t_2 (Q_1 Q_3 + Q_2 Q_4 + Q_4 Q_1).$$

But we have also $|G_2(\gamma)| = \sqrt{p}$, therefore we must conclude to

$$Q_1 Q_2 + Q_2 Q_3 + Q_3 Q_4 = Q_1 Q_3 + Q_2 Q_4 + Q_4 Q_1,$$

and to

$$2p = 2 \sum_1^4 Q_h^2 - \sum_{h,k} Q_h Q_k .$$

If we now put

$$Q_1 - Q_4 = X, \quad Q_2 - Q_3 = Y, \quad Q_1 + Q_4 - Q_2 - Q_3 = V$$

we have

$$G_1(\gamma) + G_1(\gamma^{-1}) - G_2(\gamma) - G_2(\gamma^{-1}) =$$

$$= (t_1 - t_2)\, V = 2\sqrt{p}\,(\cos\theta_1 - \cos\theta_2)\ , \quad \left|\frac{V}{5}\right| < \frac{4\sqrt{p}}{5\sqrt{5}} \qquad (13)$$

and consequently

$$S_1^2 - 4 S_2 = 5V^2 \quad \dots\dots\dots\dots \quad (14)$$

a formula showing that S_2 is an integer and that S_1 and V are both odd or both even.

Concerning the product

$$S_1 V = \frac{4p}{t_1 - t_2} (\cos^2\theta_1 - \cos^2\theta_2)$$

we may notice the inequality

$$\left|\frac{S_1 V}{5}\right| < \frac{4p}{5\sqrt{5}} \quad \ldots\ldots\ldots\ldots \quad (15)$$

and prove the relation

$$-S_1 V = X^2 + 4XY - Y^2 = (X+2Y)^2 - 5Y^2 = 5X^2 - (2X-Y)^2. \quad (16)$$

Finally we can deduce the equation

$$16p = 10X^2 + 10Y^2 + S_1^2 + 5V^2, \quad \ldots\ldots \quad (17)$$

showing that for any prime number $p = 10w + 1$ the number $16p$ can be resolved into a peculiar sum of squares, a partition the possibility of which is not easily established in another way.

Now it is possible to show that the integer V is a multiple of 5. In fact, in equation (11) the coefficients of y^2 and of y are integers. Hence we have in the first place

$$pS_1 \equiv 2 - 6p \quad , \quad (mod\ 25)$$

or

$$\begin{aligned} S_1 &\equiv -2(p+1), \quad (mod\ 25) \quad \ldots\ldots \quad (18) \\ S_1 &\equiv +1 \quad , \quad (mod\ 5). \end{aligned}$$

But we have also

$$pS_2 \equiv 1 - 6p + p^2 - 2p\, S_1, \quad (mod\ 125)$$

leading to

$$S_2 \equiv -1 \quad , \quad (mod\ 5)$$

and by combining the congruencies we will find

$$p^2(S_1^2 - 4S_2) = 5p^2V^2 \equiv 4(1-p)^3 \equiv 0 \quad , \quad (mod\ 25)$$

or

$$V \equiv 0 \quad , \quad (mod\ 5).$$

Therefore we have from (16)

$$X + 2Y \equiv 0 \quad , \quad 2X - Y \equiv 0 \quad (mod\ 5)$$

and as

$$5(X^2 + Y^2) = (X+2Y)^2 + (2X-Y)^2,$$

$\frac{X^2 + Y^2}{5}$ is also an integer.

In order to show, how the number η is connected with the other numbers, we remark that

$$\begin{aligned} \{G_1(\gamma) - G_1(\gamma^{-1})\}\{G_2(\gamma) - G_2(\gamma^{-1})\} &= -4p \sin\theta_1 \sin\theta_2 = \\ &= -(t_1 - t_2)(X^2 - XY - Y^2) \end{aligned}$$

and multiplying by

$$2\sqrt{p}(\cos\theta_1 - \cos\theta_2) = (t_1 - t_2)\, V,$$

we get

$$\left.\begin{aligned} 8p\sqrt{p}\sin\theta_1\sin\theta_2(\cos\theta_1-\cos\theta_2)=\eta=5V(X^2-XY-Y^2)= \\ =\frac{5V}{4}\{(2X-Y)^2-5Y^2\}\end{aligned}\right\} \cdot (19)$$

and so it is evident that $\frac{\eta}{25V}$ is an integer.

An useful relation follows from the identity

$$5(X^2+Y)^2-(X^2+4XY-Y^2)^2=4(X^2-XY-Y^2)^2.$$

In fact, it shows that the three integers $\frac{X^2+Y^2}{5}$, $\frac{S_1V}{5}$ and $\frac{\eta}{25V}$ are connected by the equation

$$5\left(\frac{X^2+Y^2}{5}\right)^2-\left(\frac{S_1V}{5}\right)^2=4\left(\frac{\eta}{25V}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad (20)$$

We must now pay attention to some results of LEGENDRE [1]) concerning the numbers of the form (16)

$$-S_1V=X^2+4XY-Y^2=(X+2Y)^2-5Y^2=5X^2-(2X-Y)^2.$$

The prime factors occurring in such a number can be arranged in two separate classes. The first class contains, besides a possible prime factor 5, only prime factors of the form $10x\pm1$; all factors of this class may occur raised to any power. In the second class we may find the factor 2 and possibly certain odd prime factors q different from 5 and not of the form $10x\pm1$. Their exponents necessarily are even. Hence we will have in general

$$-S_1V=X^2+4XY-Y^2=4^a\,5^b\,H^2(10y\pm1),$$

where the odd integer H is a product of prime numbers q, and $10y\pm1$ the product of the prime numbers $10x\pm1$. Moreover, we must observe that whenever a certain factor q occurs in $-S_1V$ both numbers X and Y must have this factor in common. A single factor 4 will be found in $-S_1V$, when X and Y are both odd, and only when both numbers are even the exponent a can exceed unity.

It will appear from (17) that S_1 and V cannot have a common factor q and as we have $V\equiv0, S\equiv1\ (mod\ 5)$, we must conclude that

$$S_1=4^{a_1}H_1^2(10y_1\pm1),$$
$$V=4^{a_2}\,5^b\,H_2^2(10y_2\pm1),$$

where H_1 and H_2 are prime to each other. It is possible to prove that the exponents a_1 and a_2 cannot be both greater than unity, it being clear besides that the supposition $a_1=0$ involves $a_2=0$ and inversely.

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[1]) Essai sur la théorie des nombres, Table III.

3. The preceding considerations concerning the integers S_1, S_2, V, η, X and Y enable us to ascertain their values and to calculate the coefficients of the equation of the periods for a given prime number $p = 10w + 1$, as I now will proceed to show. Let the prime number be $p = 1091$, then we begin with examining the possible values of S_1. From (18) we have

$$S_1 \equiv -9 \qquad (mod\ 25)$$

and from (10)

$$|S_1| < 133;$$

therefore we must consider as possible values the numbers

-109, -84, -59, -34, -9, $+16$, $+41$, $+66$, $+91$, $+116$.

But any factor q and also the factor 2 occurring in S_1 must have an even exponent, and so we are lead to discard the numbers -84, -34, $+$ $+66$, $+91$. For each of the remaining possible values of S_1 we examine the corresponding possible values of V. According to (13) and to (15) we have

$$\left|\frac{V}{5}\right| < 12 \quad , \quad \left|\frac{S_1 V}{5}\right| < 390,$$

and remembering that S_1 and V are both odd or both even, that they have no common factor q, and that a possible factor q in V and also the factor 2 always is raised to an even power, we evidently have to retain only the suppositions written down in the following scheme, that also indicates the corresponding values of $\frac{X^2 + Y^2}{5}$, as given by (17).

S_1	$\left\lvert\frac{V}{5}\right\rvert$	$\frac{X^2+Y^2}{5}$
-109	1	109
-59	1, 5	277, 217
-9	1, 5, 11	345, 285, 45
$+16$	4	304
$+41$	1, 5, 9	313, 253, 113

Examining the values found for $\frac{X^2 + Y^2}{5}$, we eliminate again several suppositions by observing that $\frac{X^2 + Y^2}{5}$ cannot contain a prime factor of the form $4n - 1$ with an odd exponent. Hence the numbers

$217 = 7 \times 31$, $345 = 3 \times 5 \times 23$, $285 = 5 \times 3 \times 19$,
$304 = 2^4 \times 19$, $253 = 11 \times 23$

are impossible as values of $\frac{X^2+Y^2}{5}$, so that only five suppositions are left. These are tried by means of the equation (20)

$$5\left(\frac{X^2+Y^2}{5}\right)^2-\left(\frac{S_1V}{5}\right)^2=4\left(\frac{\eta}{25V}\right)^2$$

and we find that only two of them lead to a perfect square as value of $\left(\frac{\eta}{25V}\right)^2$. Hence we have only to consider the following suppositions

S_1	$\frac{V}{5}$	$\frac{X^2+Y^2}{5}$	$4\left(\frac{\eta}{25V}\right)^2$
-109	1	109	4.109^2
-9	11	45	$4.\ 9^2$

and the evaluation of the positive or negative integers X and Y will enable us to decide between these two.

The first supposition leads to

$$X^2+Y^2=545,$$
$$X^2+4XY-Y^2=\pm 545.$$

Whatever sign we adopt in the second equation, there are no integer values of X and Y satisfying both equations and the first supposition is therefore to be rejected.

The second supposition leads to

$$X^2+Y^2=225,$$
$$X^2+4XY-Y^2=\pm 495.$$

From these equations integer values for X and Y can be found. Omitting not integer solutions, we find

$$V=+55\ \begin{cases}X=\pm 12,\\ Y=\pm\ 9,\end{cases}\quad\text{or}\quad V=-55\ \begin{cases}X=\pm\ 9,\\ Y=\mp 12,\end{cases}$$

but whatever system of solutions we make use of, we get from (14) and (19)

$$S_1=-9,\quad S_2=-3761,\quad \eta=-125.99.$$

Hence the second supposition is to be retained, and the required quintic equation of the periods (11) becomes

$$0=y^5+y^4-436\,y^3+131\,y^2+42453\,y-3015 \qquad . \quad . \quad (21)$$

At the same time we are now able to give the complete algebraic solution of this equation. From (12) we have

$$9=Q_1+Q_2+Q_3+Q_4$$

and this equation is to be combined with

$$\left\{\begin{array}{l} +55=Q_1+Q_4-Q_2-Q_3, \\ \pm 12=Q_1-Q_4, \\ \pm\ \ 9=Q_2-Q_3, \end{array}\right. \qquad \text{or with} \quad \left\{\begin{array}{l} -55=Q_1+Q_4-Q_2-Q_3, \\ \pm\ \ 9=Q_1-Q_4, \\ \mp 12=Q_2-Q_3. \end{array}\right.$$

In this way we find four systems of integer values for the numbers Q_1, Q_2, Q_3 and Q_4, corresponding with the four sums $G_1(\gamma)$, $G_1(\gamma^2)$, $G_1(\gamma^3)$ and $G_1(\gamma^4)$. In one of these sums for example in $G_1(\gamma^k)$, the numbers $Q_1+Q_4-Q_2-Q_3$ and Q_1-Q_4 will be positive. Selecting this sum, we will find

$$Q_1=22, \quad Q_2=-7, \quad Q_3=-16, \quad Q_4=10$$

and may then write

$$G_1(\gamma^k)=22\gamma-7\gamma^2-16\gamma^3+10\gamma^4,$$

where k is one of the numbers 1, 2, 3 or 4. Now γ^k as well as γ itself is a root of (9), therefore we can solve equation (21) by putting

$$f(\alpha, \gamma^k)^5=1091\, G_1(\gamma^k)^2\, G_1(\gamma^{2k}),$$

and then by writing

$$5y+1=f(\alpha, \gamma^k)+\frac{1091}{f(\alpha, \gamma^k)}+\frac{f(\alpha, \gamma^k)^2}{G_1(\gamma^k)}+\frac{1091\, G_1(\gamma^k)}{f(\alpha, \gamma^k)^2}.$$

From the method applied to the treatment of the particular case $p=1091$ it will be evident that in general a finite number of trials will suffice to ascertain the numerical values of S_1, S_2 and η, to write down the equation of the periods and to arrive at its algebraic solution.

4. In the case $p=6w+1$, $\nu_1=6$ the primitive root γ of (2) is one of the roots of the equation

$$\gamma^2-\gamma+1=0 \quad . \ . \ . \ . \ . \ . \ . \ . \quad (22)$$

and according to the general formulae (5), (7) and (8) we have

$$G_0(\gamma)=-1), \quad G_3(\gamma)=(-1)^w G_2(\gamma), \quad G_4(\gamma)=(-1)^w G_1(\gamma), \quad G_5(\gamma)=(-1)^{w-1} p,$$

$$f(\alpha, \gamma)^6=(-1)^w p\, G_1^2(\gamma)\, G_2^2(\gamma),$$

$$6y+1=f(\alpha, \gamma)+\frac{f(\alpha, \gamma)^2}{G_1(\gamma)}+\frac{f(\alpha, \gamma)^3}{G_1(\gamma)\, G_2(\gamma)}+\frac{(-1)^w f(\alpha, \gamma)^4}{G_1(\gamma)\, G_2(\gamma)^2}+\frac{f(\alpha, \gamma)^5}{G_1(\gamma)^2\, G_2(\gamma)},$$

or

$$6y+1=\left\{f(\alpha, \gamma)+\frac{(-1)^w p}{f(\alpha, \gamma)}\right\}+\left\{\frac{f(\alpha, \gamma)^2}{G_1(\gamma)}+\frac{p\, G_1(\gamma)}{f(\alpha, \gamma)^2}\right\}+i^w\sqrt{p},$$

the latter equation showing that the six periods all are real when w is even and that they are complex when w is odd.

Eliminating $f(\alpha, \gamma)$ the equation of the periods presents itself in the form

$$\left.\begin{aligned}
&i^w\sqrt{p}\Big[108y^2+18y\{2+G_2(\gamma)+G_2(\gamma^{-1})\}+3-p(3+2(-1)^w)+\\
&\qquad\qquad+3(-1)^w p\left\{\frac{G_1(\gamma)}{G_2(\gamma)}+\frac{G_2(\gamma)}{G_1(\gamma)}\right\}+3\{G_2(\gamma)+G_2(\gamma^{-1})\}+\\
&+\{G_1(\gamma)\,G_2(\gamma)+G_1(\gamma^{-1})\,G_2(\gamma^{-1})\}\Big]=216y^3+108y^2+18y(1-p)+\\
&+1-3p+3(-1)^w p\{G_2(\gamma)+G_2(\gamma^{-1})\}-3p\{G_1(\gamma)+G_1(\gamma^{-1})\}-\\
&\qquad\qquad\qquad\qquad-(-1)^w p\left\{\frac{G_2^2(\gamma)}{G_1(\gamma)}+\frac{G_2^2(\gamma^{-1})}{G_1(\gamma^{-1})}\right\}
\end{aligned}\right\}\quad(23)$$

and the following considerations lead to the evaluation of the coefficients. Supposing $k\mu \not\equiv 0$, $k(\mu+1) \not\equiv 0 \pmod 6$ we have (4)

$$G_\mu(\gamma^k)=\sum_{r_1,r_2}\gamma^{k(r_1+\mu r_2)}=A+B\gamma,$$

and as $|G_\mu(\gamma_k)|=\sqrt{p}$, we get

$$\left.\begin{aligned}
p=A^2+AB+B^2=(A+B)^2-AB=(A-B)^2+3AB,\\
4p=(2A+B)^2+3B^2=(2B+A)^2+3A^2,
\end{aligned}\right\}\quad(24)$$

from which equations we infer that the factor p does not occur in one of the integers A, B, $A+B$, $A-B$, $2A+B$ and $2B+A$. Likewise we observe that in the equations

$$(A+B\gamma)^2=(A^2-B^2)+\gamma B(2A+B)=U_2+V_2\gamma,$$
$$p^2=U_2^2+U_2V_2+V_2^2$$

the integers U_2 and V_2 in this respect are similar to A and B and that, if we write

$$(A+B\gamma)^4=U_4+V_4\gamma,$$

again neither U_4 nor V_4 is a multiple of p. Now on the ellipse

$$p=x^2+xy+y^2$$

there always lie twelve points both coordinates of which are integers, and by a finite number of trials their coordinates can be found. Selecting one of these points with the coordinates A, B, we may assert that, always in the supposition $k\mu\not\equiv 0$, $k(\mu+1)\not\equiv 0 \pmod 6$, any sum $G_\mu(\gamma^k)$ can be written in the form $\beta^h(A+B\beta)$, where β is either γ or γ^{-1}, if we only give the right value to the exponent h. So for instance we have

$$G_1(\gamma^2)=\beta^h(A+B\beta),\quad G_1(\gamma)=\beta_1^{h_1}(A+B\beta_1),\quad G_2(\gamma)=\beta_2^{h_2}(A+B\beta_2)$$

and we must prove that β, β_1 and β_2 represent one and the same root of (22). For this purpose we consider the equation

$$G_1(\gamma)\,G_1(\gamma^2)=(-1)^w\,G_2(\gamma)^2,\quad\ldots\ldots\ldots\quad(25)$$

immediately derived from (6), which we write in the form

$$\beta_1^{h_1}\beta^h(A+B\beta_1)(A+B\beta)=(-1)^w\,\beta_2^{2h_2}(A+B\beta_2)^2.$$

The supposition that β_1 and β were different roots of equation (22) would lead to

$$\beta_2^\sigma p = (A + B\beta_2)^2 = U_2 + V_2\beta$$

and that is impossible since neither U_2 nor V_2 has the factor p. Further, β_1 and β being equal, they cannot differ from β_2 because in that case we would arrive at

$$\beta^{h+h_1}(A + B\beta)^2 = (-1)^w \beta_2^{2h}(A + B\beta_2)^2,$$

$$\beta_2^\sigma p^2 = (A + B\beta_2)^4 = U_4 + V_4\beta_2,$$

and again that would be impossible.

In the first place I proceed to the evaluation of h by deducing the equation of the periods of order $\frac{p-1}{3}$. Remembering that γ still denotes one of the roots of (22), we have according to (5), (7) and (8)

$$G_0(\gamma^2) = -1, \quad G_2(\gamma^2) = -p$$

$$f(\alpha, \gamma^2)^3 = p\, G_1(\gamma^2)$$

$$3y + 1 = f(\alpha, \gamma^2) + \frac{f(\alpha, \gamma^2)^2}{G_1(\gamma^2)} = f(\alpha, \gamma^2) + \frac{p}{f(\alpha, \gamma^2)}$$

and eliminating $f(\alpha, \gamma^2)$ the required equation becomes

$$y^3 + y^2 + \frac{1}{3}\, y\,(1-p) + \frac{1}{27}\left[1 - 3p - p\{G_1(\gamma^2) + G_1(\gamma^{-2})\}\right] = 0. \quad (26)$$

Here the last coefficient is an integer, hence we have

$$1 - 3p - p\{G_1(\gamma^2) + G_1(\gamma^{-2})\} \equiv 0 \qquad (mod\ 27)\ .\ .\ .\ (27)$$

and this congruency will give the value of h as soon as p is a known prime number. In fact, we may write

$$\beta^h = L + M\beta$$

and the corresponding values of h, L and M must be those of the following scheme

h	L	M
0	$+1$	0
1	0	$+1$
2	-1	$+1$
3	-1	0
4	0	-1
5	$+1$	-1.

In this way we get

$$G_1(\gamma^2) + G_1(\gamma^{-2}) = L\,(2A + B) + M\,(A - B)$$

and then from (27) the congruency

$$1 - 3p - pL\,(2A + B) - pM\,(A - B) \equiv 0, \qquad (mod\ 27)$$

by which the exponent h is readily determined. A second congruency will determine the exponent h_2. In fact, the equation of the periods of order $\frac{p-1}{6}$ (23), when rationalised, must have integer coefficients, and that requires

$$2 + G_2(\gamma) + G_2(\gamma^{-1}) \equiv 0 \qquad (mod\ 6)\ .\ .\ .\ .\ .\ \ (28)$$

or

$$2 + L\,(2A + B) + M\,(A - B) \equiv 0. \qquad (mod\ 6)$$

This congruency in connexion with the preceding scheme will give the value of h_2 and finally h_1 is found from the congruency

$$h_1 + h \equiv 3w + 2h_2 \qquad (mod\ 6)\ .\ .\ .\ .\ .\ \ (29)$$

directly derived from (25).

Now all the coefficients of (23) are known, and we have only to rationalise this equation, to obtain the equation of the periods of order $\frac{p-1}{6}$.

5. As an example I will treat the case $p = 1063$. Equation (24) becomes

$$(2A + B)^2 = 4252 - 3B^2,$$

and after a few trials we find ± 3, ± 31 and ± 34 as possible values of B. Giving, for instance, B the value 34 and adopting -3 as one of the two corresponding values of A, we may write

$$G_1(\gamma^2) = \beta^h\,(-3 + 34\,\beta).$$

Hence the congruency (27) becomes

$$-2 - 10\,L - 8\,M \equiv 0 \qquad (mod\ 27)$$

and considering the scheme, we find that we have necessarily $L = -1$, $M = +1$, $h = 2$. Having thus established the equation

$$G_1(\gamma^2) = \beta^2\,(-3 + 34\beta),$$

we have also the result

$$G_1(\gamma^2) + G_1(\gamma^{-2}) = -65,$$

and by the way we have found that the equation (26) of the periods of order $\frac{p-1}{3}$ is

$$y^3 + y^2 - 354\,y + 2441 = 0.$$

The congruency (28) now takes the form

$$2 + 28\,L - 37\,M \equiv 0 \qquad (mod\ 6),$$

and we necessarily have $L = 1,\ M = 0,\ h_2 = 0,$

$$G_2(\gamma) = (-3 + 34\,\beta).$$

Hence from (29) we get $h_1 = 1$

$$G_1(\gamma) = \beta\,(-3 + 34\,\beta)$$

and then successively

$$G_1(\gamma) + G_1(\gamma^{-1}) = -37, \qquad G_2(\gamma) + G_2(\gamma^{-1}) = 28,$$

$$\frac{G_1(\gamma)}{G_2(\gamma)} + \frac{G_1(\gamma^{-1})}{G_2(\gamma^{-1})} = 1, \qquad G_1(\gamma\,)G_2(\gamma) + G_1(\gamma^{-1})\,G_2({}^{-1}) = -2099,$$

$$\frac{G_2^2(\gamma)}{G_1(\gamma)} + \frac{G_2^2(\gamma^{-1})}{G_1(\gamma^{-1})} = 65.$$

Substituting these results in (23) we obtain

$$i\sqrt{1063}\left(\frac{1}{2}\,y^2 + \frac{5}{2}\,y - 29\right) = \left(y^3 + \frac{1}{2}\,y^2 - \frac{177}{2}\,y + 438\right)$$

and then by squaring we find

$$0 = y^6 + y^5 + 89\,y^4 + 3445\,y^3 - 15913\,y^2 - 231661\,y + 1085827$$

as the equation of the periods of order $\frac{p-1}{6}$, an equation all roots of which are complex. Solving algebraically, we have at once

$$f(\alpha,\gamma)^6 = -1063\,\beta^2\,(-3 + 34\,\beta)^4 = -1063\,\beta^2\,R^4,$$

$$6y + 1 = f(\alpha,\gamma) - \frac{1063}{f(\alpha,\gamma)} + \frac{f^2(\alpha,\gamma)}{\beta R} + \frac{1063\,\beta R}{f^2(\alpha,\gamma)} + \frac{f^3(\alpha,\gamma)}{\beta R^2}.$$

It will be found that the equation of the periods of order $\frac{p-1}{4}$ may be derived in a quite similar way.

Chemistry. — "*Equilibria in systems, in which phases, separated by a semi-permeable membrane*" IX. By Prof. F. A. H. SCHREINEMAKERS.

(Communicated at the meeting of May 30, 1925).

Two ternary systems.

We take an osmotic equilibrium:

$$E = E_1 \mid E_2$$

in which the separate parts E_1 and E_2 are both ternary systems. Of course they have always one component in common, viz. the diffusing substance W (water); both the other components can be either the same or different. Consequently we distinguish three cases:

1. There are three common components; then each of the separate systems contains f.i. the substances $W+X+Y$.

2. There are only two common components; then the one system contains f.i. the substances $W+X+Y$ and the other system the substances $W+X+Z$.

3. There is only one common component, viz. the diffusing substance W. Then the one system contains f.i. the substances $W+X+Y$ and the other system the substances $W+Z+U$.

In previous communications we have discussed already several examples of the first case; now we shall consider some examples of the other cases.

Both the systems contain two common components.

We represent the phases of the system E_1, which contains the components $W+X+Y$ by points of a triangle WXY; those of a system E_2, which contains the components $W+X+Z$ by points of a triangle WXZ. In fig. 1 both triangles are placed against one another with the side WX, they are united in a single diagram.

It appears from the figure that we have assumed that in the system $W+X+Y$ both the components X and Y occur as solid phases; in the system $W+X+Z$, however, only the solid substance X. The saturation-curves of those systems are represented by the curves ab, db and dg. In accordance with the previous the arrows indicate the direction in which the O.W.A. of the saturated liquids increases.

We consider, just as formerly, the osmotic equilibrium:

$$E = E_1 \mid E_2 \quad \dots\dots\dots\dots \quad (1)$$

again at constant temperature and under constant pressure; we assume again that the pressure P_1 of system E_1 is equal to the pressure P_2 of system E_2. Then follows from the membrane-phase-rule (Comm. VII):

$$\text{number of freedoms} = 3 + 3 - (r_1 + r_2) - 1 = 5 - (r_1 + r_2) \quad . \quad (2)$$

in which r_1 indicates the number of phases of E_1 and r_2 that of E_2.

In the osmotic equilibrium:

$$E = L_1 \mid L_2 \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (3)$$

L_1 and L_2 represent two liquids. If we take a binary liquid L_q, represented by point q of the side WX, then all liquids of the system $W+X+Y$, which are isotonic with L_q are represented by points of the isotonic curve qq_1; all liquids of the system $W+X+Z$ by an isotonic curve qq_2.

Consequently the two isotonic curves qq_1 and qq_2 start from point q; the one is situated in triangle WXY, the other in triangle WXZ. Each liquid of curve qq_1, therefore, is not only isotonic with each other liquid of this curve, but also with all liquids of curve qq_2.

Generally, therefore, with each liquid, which contains the components $W+X+Y$ is isotonic not only a series of liquids with the same components, but also a series of liquids with the components $W+X+Z$.

Consequently two isotonic curves start from each point of the side WX; we call them conjugated isotonic curves; some of them are dotted in fig. 1. Liquid r_1 is isotonic, therefore, with the liquids L_t, L_d, L_v and L_{r_2}; liquid a with the liquids L_c', L_e, L_u and L_h; etc. The binary $W+Y$-containing liquid r_1 is isotonic, therefore, with the binary $W+Z$-containing liquid r^2; etc.

Also follows from the above considerations that the osmotic equilibrium (3) has three freedoms. If we take f.i. for L_1 a definite liquid, f.i. L_t, then two freedoms disappear, consequently L_2 has left still one freedom; it is represented by a point of the conjugated isotonic curves dr_1 and dr_2. We are able to represent this (compare f. i. previous communications) by:

$$E = L_t \mid L\,(1 - dr_1 - dr_2) \quad . \; . \; . \; . \; . \; . \; . \; . \quad (4)$$

from which it appears that the liquid L has one freedom and is represented by a point of the conjugated curves dr_1 and dr_2.

If we represent the thermodynamical potential and the composition of liquid L_1 by

$$\zeta_1 \text{ and } x_1X + y_1Y + (1 - x_1 - y_1)W \quad . \; . \; . \; . \; . \quad (4a)$$

and those of L_2 by:

$$\zeta_2 \text{ and } x_2X + z_2Z + (1 - x_2 - z_2)W \quad . \; . \; . \; . \; . \; . \quad (4b)$$

then is valid for equilibrium (3) the equation:

$$\zeta_1 - x_1\frac{\partial\zeta_1}{\partial x_1} - y_1\frac{\partial\zeta_1}{\partial y_1} = \zeta_2 - x_2\frac{\partial\zeta_2}{\partial x_2} - z_2\frac{\partial\zeta_2}{\partial z_2} \quad . \quad . \quad . \quad . \quad (5)$$

As in (5) four variables occur, it also follows from this that the osmotic equilibrium (3) has three freedoms. This also follows at once from (2), if we put herein $r_1 = 1$ and $r_2 = 1$.

If we bring in osmotic contact the liquids L_s and L_u (fig. 1) then, as the O.W.A. of L_u is larger than that of L_s, water shall diffuse from L_s towards L_u. Therefore, liquid L_s moves along the straight line Ws, away from point W, liquid L_u along the straight line uW towards W. This diffusion continues till both liquids have got the same O.W.A., therefore, till both come on conjugated isotonic curves. When this is f. i. the case on the curves dr_1 and dr_2, then is formed the osmotic equilibrium:

$$E = L_t \mid L_v \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

As we shall deduce further, it depends not only on the composition, but also on the ratio of the quantities of the original liquids L_s and L_u, on which isotonic curves they will get the same O.W.A.

In order to show this, we take n_1 quantities of L_s with the composition (4^a) and n_2 quantities of L_u with the composition (4^b).

We assume that from this arise, n_1' quantities of a liquid $L_{s'}$ with the composition:

$$x_1'X + y_1'Y + (1 - x_1' - y_1')W$$

and n_2' quantities of $L_{u'}$ with the composition:

$$x_2'X + z_2'Z + (1 - x_2' - z_2')W.$$

If we express that the total quantity of the diffusing substance W remains constant and that also the quantity of each of the not-diffusing substances X, Y and Z remains constant on each of the sides of the membrane, then we find the equations:

$$\left.\begin{array}{c} n_1' + n_2' = n_1 + n_2 \\ n_1'x_1' = n_1x_1 \qquad n_1'y_1' = n_1y_1 \qquad n_2'x_2' = n_2x_2 \qquad n_2'z_2' = n_2z_2 \end{array}\right\} \quad (7)$$

From this follows by elimination of n_1' and n_2':

$$\frac{x_1'}{y_1'} = \frac{x_1}{y_1} \qquad \frac{x_2'}{x_2} = \frac{z_2'}{z_2} \qquad 1 + \frac{n_2}{n_1} = \frac{x_1}{x_1'} + \frac{n_2}{n_1} \cdot \frac{x_2}{x_2'} \quad . \quad . \quad . \quad (8)$$

The first one of those equations expresses that the liquid $L_{s'}$ proceeds along the line Ws, the second one expresses that the liquid $L_{u'}$ proceeds along the line Wu; it appears from the third one that the composition depends also on the ratio $n_2:n_1$.

The equations (8) are valid for each arbitrary moment of the diffusion; those three equations contain four unknowns. When the equilibrium takes place, then the variables x'_1, y'_1, x'_2 and z'_2 have to satisfy also an equation, which is deduced from (5) by giving in this an accent to each of the variables. Then we have four equations with four unknowns so that the composition of both the liquids after the diffusion is defined completely.

We imagine to be drawn in fig. 1 on the line su a point k, which divides this line into two parts, which are defined by:

$$sk:uk=n_2:n_1 \quad \ldots\ldots\ldots \quad (9)$$

We may imagine that this point k represents the complex of n_1 quantities of L_s and n_2 quantities of L_u.

During the diffusion the liquid $L_{s'}$ is situated anywhere in a point s' on the line st (between s and t), the liquid $L_{u'}$ anywhere in a point u' on the line uv (between u and v). We are able now to show that the conjugation-line $s'u'$ goes perpetually through the point k. Consequently this must also be the case with the line tv.

The above should be clear without more, if the liquids u' and s' belong to the same ternary system; here, however, we have two different ternary

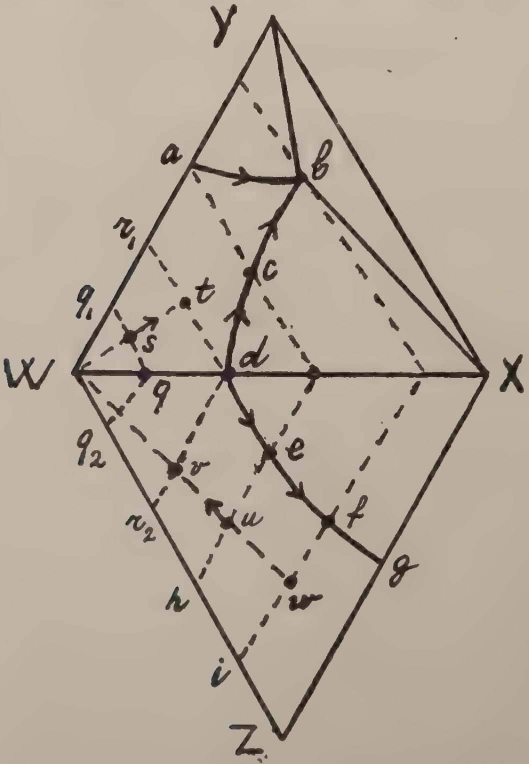

Fig. 1

systems which form together a quaternary system. The liquids under consideration are situated in two of the four triangles, which limit the tetrahedron $WXYZ$, which should represent in space this quaternary equilibrium. Consequently the complex k is situated really in space within the tetrahedon and not in the plane of fig. 1, which represents only two side-planes of the tetrahedron, drawn in a same plane.

If, however, we consider fig. 1 as a perspective projection of the tetrahedon $WXYZ$, then follows from this at once the property above mentioned; we are able to deduce this still also in the following way. From fig. 1 viz. follows that we may put:

$$Ws = \alpha x_1 \qquad Ws' = \alpha x_1' \qquad Wu = \beta x_2 \qquad Wu' = \beta x_2' \quad , \quad (10)$$

in which α and β have definite values. If we take the line Wu as X-axis and the line Wt as Y-axis of a system of coördinates, then is:

$$\frac{X}{\beta x_2}+\frac{Y}{\alpha x_1}=1 \qquad \text{the equation of the line } su$$

$$\frac{X}{\beta x_2'}+\frac{Y}{\alpha x_1'}=1 \qquad \text{,, \quad ,, \quad ,, \quad ,, \quad ,, } \; s'u'$$

Calculating from this the X or the Y of the point of intersection k of those two lines, then we easily find:

$$sK : uK = \left(\frac{x_1}{x_1'}-1\right) : \left(1-\frac{x_2}{x_2'}\right) \; . \; . \; . \; . \; . \; . \quad (11)$$

With the aid of (7) we find from this the equation (9) so that the property above mentioned is proved.

In fig. 1, therefore, k shall be the point of intersection of the lines su and tv; the liquids s' and u', which occur during the diffusion, are situated, therefore, in such a way that the line $s'u'$ goes through the point k.

We now consider the osmotic system:

$$X + L_d \;\vdots\; L \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (12)$$

of fig 1. As L_d is the liquid, which is formed when the solid substance X takes a little water in, L_d has the same O.W.A. as the solid substance X. In a similar way and with the same way of representation as previously (compare f.i. Comm. VII) we find:

$$E = X + L_d \;\vdots\; L\,(1 - dr_1 - dr_2) \; . \; . \; . \; . \; . \; . \; . \quad (13)$$

$$X \overset{|}{\underset{|}{\leftarrow}} L\,(2 - Wdr_1 - Wdr_2) \; . \; . \; . \; . \; . \; . \; . \quad (14)$$

$$X \;\vdots\; L\,(2 - dr_1\,YX - dr_2\,ZX) \; . \; . \; . \; . \; . \quad (15)$$

Limiting ourselves to stable liquids (viz. only to the unsaturated and saturated liquids, excluding the supersaturated liquids) then (15) passes into:

$$X \mid L\,(2 - dr_1\,ab - dr_2\,Zg) \quad . \; . \; . \; . \; . \; . \; . \; . \quad (16)$$

(13) indicates that the liquid L, which is isotonic with the liquid L_d, has one freedom and is represented by points of the curves dr_1 and dr_2.

(14) means that the liquids from which water diffuses towards the solid substance X, have two freedoms and are represented by points of the regions Wdr_1 and Wdr_2.

(16) means that the liquids, which can be in osmotic contact with the solid substance X, without anything happening, have two freedoms and are represented by points of the region $dr_1\,a\,b$ or $dr_2\,Z\,g$.

We may express this also in the following way:

the O.W.A. of the solid substance X is equal to that of the liquids of curves dr_1 and dr_2, larger than that of the liquids of the regions Wdr_1 and Wdr_2 and smaller than that of the liquids of the regions $dr_1\,a\,b$ and $dr_2\,Z\,g$.

Let us take now the osmotic system:

$$Y + L_a \mid L \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (17)$$

in which L_a (fig. 1) is the liquid, which arises, when the solid substance Y takes a little of the diffusing substance W. We now find:

$$E = Y + L_a \mid L\,(1 - ac - eh) \quad . \; . \; . \; . \; . \; . \; . \quad (18)$$

$$Y \leftarrow\!\mid\! L\,(2 - Wdca - Wdeh) \quad . \; . \; . \; . \; . \; . \; . \quad (19)$$

$$Y \mid L\,(2 - acb - eh\,Zg) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (20)$$

This means: the O.W.A. of the solid substance Y is equal to that of the liquids of curves ac and eh, larger than that of the liquids of the regions $Wdca$ and $Wdeh$ and smaller than that of the liquids of the regions acb and $ehZg$.

Consequently here a series of liquids exists, which do not contain the substance Y (viz. of curve eh) and yet they have the same O.W.A. as the solid substance Y. Also there are now two liquids, saturated with the solid substance X, which have also the same O.W.A. as the solid substance Y. One of them (viz. liquid c) contains the substance Y, the other one (viz. liquid e), however, does not contain the substance Y.

In the osmotic system:

$$X + Y + L_b \mid L \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad (21)$$

L_b is the liquid which arises, when solid $X+Y$ takes a little water in. Limiting ourselves again to stable states, then we find:

$$E = X + Y + L_b \;\overset{|}{\underset{|}{}}\; L\,(1-fi) \quad \ldots\ldots\ldots \tag{22}$$

$$X + Y \overset{|}{\underset{|}{\leftarrow}} L\,(2-Wdba-Wdfi) \quad \ldots \tag{23}$$

$$X + Y \;\overset{|}{\underset{|}{}}\; L\,(2-fi\,Zg) \quad \ldots\ldots\ldots \tag{24}$$

Consequently there is only one liquid, containing $X+Y$, viz. L_b, which has the same O.W.A. as the solid mixture $X+Y$; indeed there is still a series of liquids, viz. those of curve fi, which also have this same O.W.A. but those liquids do not contain the substance Y.

Also we find in the system $W+X+Y$ no liquids with a greater O.W.A. than those of the solid mixture $X+Y$; we find them in the region $fiZg$ of the system $W+X+Z$.

If we bring in osmotic contact L_c and L_u then nothing happens; both liquids are situated viz. on conjugated isotonic curves and have, therefore, the same O.W.A.

If we bring in osmotic contact L_c and L_v then, as the O.W.A. of L_c is greater than that of L_v, the inversion:

$$L_c \overset{|}{\underset{|}{\leftarrow}} L_v$$

shall occur; the saturated liquid L_c shall become unsaturated, therefore.

If we bring in osmotic contact L_c and L_w, then water shall diffuse from L_c towards L_w; consequently we have:

$$L_c \overset{|}{\underset{|}{\rightarrow}} L_w\,.$$

Consequently L_c shall pass into a liquid between c and b, with separation of the solid substance X.

If we bring in osmotic contact solid $X+Y$ with L_v, then, as the O.W.A. of the solid $X+Y$ is greater than that of liquid L_v, water shall diffuse from the liquid towards $X+Y$. Consequently we get the inversion:

$$X + Y \overset{|}{\underset{|}{\leftarrow}} L_v\,.$$

When a sufficient quantity of $X+Y$ is present, then at last is formed the osmotic equilibrium:

$$E = X + Y + L_b \;\overset{|}{\underset{|}{}}\; L_w\,.$$

Both the systems contain one common component only.

We now take the osmotic equilibrium:

$$E = E_1 \,\vert\, E_2$$

in which the separate parts have only the diffusing substance W in common. We shall represent the phases of E_1 with the components $W+X+Y$ by points of triangle WXY (fig. 2); those of system E_2 with the components $W+Z+U$ by points of triangle WZU (fig. 3).

We assume, as is apparent from those two diagrams, that in the one system the components X and Y occur as solid phases, in the other system the component U. The arrows on the saturation-curves *ci*, *hi* and *em* indicate the direction in which the O.W.A. of the saturated solutions increases.

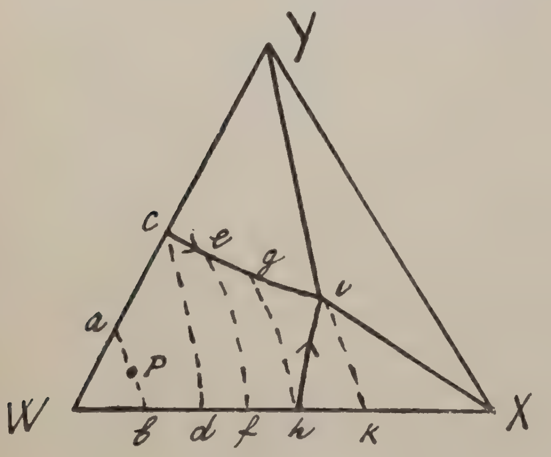

Fig. 2.

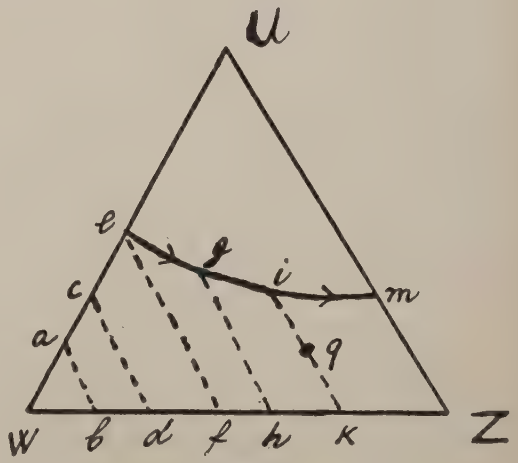

Fig. 3.

We now draw in fig. 2 an isotonic curve *ab*; now we are able to draw also in fig. 3 an isotonic curve *ab*, the liquids of which have the same O.W.A. as the liquids of curve *ab* in fig. 2. The curves *ab* of figs. 2 and 3, which we shall call for the sake of distinction $(ab)_2$ and $(ab)_3$ are, therefore, conjugated isotonic curves.

We now assume that the dotted curves, which are indicated in both figures by the same letters, represent also conjugated isotonic curves; f.i. $(cd)_2$ and $(cd)_3$; $(ef)_2$ and $(ef)_3$; etc.

For the osmotic equilibrium:

$$E = L_2 \,\vert\, L_3$$

in which L_2 represents a liquid of fig. 2 and L_3 a liquid of fig. 3, is true the equation for equilibrium:

$$\zeta_2 - x\frac{\partial \zeta_2}{\partial x} - y\frac{\partial \zeta_2}{\partial y} = \zeta_3 - z\frac{\partial \zeta_3}{\partial z} - u\frac{\partial \zeta_3}{\partial u} \quad . \quad . \quad . \quad . \quad (25)$$

in which ζ_2 and ζ_3 represent the thermodynamical potentials of the liquids L_2 and L_3.

We now bring in osmotic contact liquid L_p [on curve $(a\,b)_2$] with liquid L_q [on curve $(i\,k)_3$]. As the O.W.A. of L_q is greater than that of L_p, water shall diffuse from L_p towards L_q. Consequently L_q moves along the line Wq starting from q towards W; L_p shifts along the line Wp further away from p. For this diffusion, which continues till both liquids reach conjugated isotonic curves, the same is true as is deduced above for fig. 1.

In the osmotic system:

$$Y + L\,(c)_2 \;\vert\; L$$

$L(c)_2$ (fig. 2) is the liquid, which arises when a little water diffuses towards the solid substance Y. Limiting ourselves again to stable states, then follows:

$$E = Y + L\,(c)_2 \;\vert\; L\,[1-(cd)_2-(cd)_3]$$

$$Y \leftarrow\!\!\!\!\mid\; L\,[2-(Wcd)_2-(Wcd)_3]$$

$$Y \;\vert\; L\,[2-(cdhi)_2-(cdZme)_3]\,.$$

This means: the O.W.A. of the solid substance Y is equal to that of the liquids of curve $c\,d$ in figs. 2 and 3; greater than that of the liquids of the region $W\,c\,d$ in figs. 2 and 3; smaller than that of the liquids of the region $c\,d\,h\,i$ in fig. 2 and of the region $c\,d\,Z\,m\,e$ in fig. 3.

If we bring in osmotic contact the solid substance Y with liquid q (which contains the components $W+U+Z$) then nothing happens, therefore; if, however, we bring in osmotic contact solid Y with a liquid of curve $(a\,b)_3$ [which contains, therefore, also the components $W+U+Z$] then Y shall flow away totally or partly.

In the osmotic system:

$$X + L\,(h)_2 \;\vert\; L$$

$L(h)_2$ (fig. 2) is the liquid which arises when the solid substance X takes a little water in.

We find from this:

$$E = X + L\,(h)_2 \;\vert\; L\,[1-(hg)_2-(hg)_3]$$

$$X \leftarrow\!\!\!\!\mid\; L\,[2-(Whgc)_2-(Whge)_3]$$

$$X \;\vert\; L\,[2-(hgi)_2-(hgmZ)_3].$$

From this appears a.o. that from all liquids of the region $Whgc$ in fig. 2 (consequently with the components $W+X+Y$) and from all liquids of the region $Whge$ in fig. 3 (consequently with the components $W+Z+U$) water will diffuse towards the solid substance X; the other liquids let X unchanged.

In similar way the reader may find also the liquids, which have either the same or smaller or greater O.W.A. than the solid substance U or the solid mixture $X+Y$.

If we bring in osmotic contact the solid substances X, Y or U or the solid mixture $X+Y$ with a liquid L, which contains the components $W+X+Y$ (fig. 2) or the components $W+Z+U$ (fig. 3) then we may distinguish several cases.

1. L is situated within the region Wcd of fig. 2 or 3.
   X, Y, U and $X+Y$ flow away totally or partly.
2. L is situated within the region $cdfe$ of fig. 2 or 3.
   X, U and $X+Y$ flow away totally or partly; Y remains unchanged.
3. L is situated within the region $efhg$ of fig. 2 or 3.
   X and $X+Y$ flow away totally or partly; Y and U remain unchanged.
4. L is situated within the region hgi of fig. 2 or within the region $hgik$ of fig. 3.
   $X+Y$ flows away totally or partly; X, Y and U remain unchanged.
5. L is situated within the region $ikZm$ of fig. 3.
   X, Y, U and $X+Y$ remain unchanged.

Leiden, Lab. of Inorg. Chemistry. *(To be continued.)*

Physics. — *"The characteristics of tungsten and the candle power of the black body"*. By C. ZWIKKER. (Communicated by Prof. P. ZEEMAN).

(Communicated at the meeting of May 30, 1925).

The importance of tungsten for the glowlamp and radio industry, as well as the scientific interest we have in an extensive knowledge of the physical properties of a material, which may be obtained in a very pure state, have lead me to a redetermination of the diverse properties of this metal at high temperatures. The final results have been collected in table I.

The temperature has been measured by an optical pyrometer method, making use of PLANCK's radiation law. The value of C_2 which occurs in this formula is taken as 14 330 μ degr. The basispoint used for the temperature scale is the melting point of gold, for which the value of 1336° KELVIN is adopted. The melting point of the sample of gold, used by me, which was kindly supplied to us by Dr. v. HETEREN, was in concordance with the goldpoint of the Physikalisch Technische Reichsanstalt.

The spectral emissivity has been determined by comparing the inner and outer brightness of a tungsten rod in which a hole had been drilled and which was raised to a high temperature by different methods. With this so determined spectral emissivity e_λ, the true temperature T has been calculated from the observed „brightness temperature" S by the relation:

$$\frac{1}{T}-\frac{1}{S}=\frac{\lambda\, 2{,}303 \log e_\lambda}{C_2}.$$

Electrical measurements have been performed by a compensation method. The standard resistance and the standard element had been verified by the P.T.R. The *resistivity* was obtained from the electrical resistance of a tungsten filament of known dimensions; the *total radiation* from the wattage input of this filament.

Corrections have been made for the cooling at the filament junctions.

The brightness given in the table is the normal candle power, expressed in International Candles. The photometric standard lamps had been standardized by the P. T. R. in 1922. The candle power was given by the P. T. R. in Hefner Candles and was converted by us to International Candles, assuming that:

$$1\ I.C. = 1{,}11\ H.C.$$

TABLE I. The characteristics of tungsten.

Temperature	Spectral emissivity $\lambda = 0.665$	Brightness temperature $\lambda = 0.665$	Colour temperature	Resistivity	Total radiation	Brightness	Thermionic emission	Rate of vaporization	Thermal conductivity	Thomson-effect
T	e_λ	S_λ	T_c	ρ	η	B	i	m	k	σ
1200	0.458	1150	1210	30.9	1.70	0.0071				
1300	0.456	1240	1312	34.0	2.70	0.0311				
1400	0.454	1331	1414	37.1	3.94	0.117				
1500	0.451	1421	1517	40.2	5.52	0.370	$0.102.10^{-6}$	$1.78.10^{-22}$		
1600	0.449	1510	1619	43.4	7.90	1.07	$0.102.10^{-5}$	$1.78.10^{-20}$		
1700	0.446	1597	1722	46.7	10.7	2.62	$0.812.10^{-5}$	$1.02.10^{-18}$		
1800	0.444	1685	1825	49.9	14.1	5.75	$0.490.10^{-4}$	38.0	1.22	—18
1900	0.442	1773	1929	53.2	18.6	12.3	$0.257.10^{-3}$	932	1.26	—20
2000	0.440	1859	2033	56.7	24.0	23.2	$0.112.10^{-2}$	$15.5\ .10^{-15}$	1.29	—22
2100	0.438	1945	2137	60.1	30.5	42.4	0.00427	214	1.31	—24
2200	0.435	2028	2242	63.5	38.2	72.0	0.0141	$2.24.10^{-12}$	1.33	—26
2300	0.432	2111	2347	66.9	47.2	116.5	0.0437	19.5	1.35	—28
2400	0.430	2193	2452	70.5	57.3	185	0.123	138	1.37	—30
2500	0.427	2275	2557	74.0	69.4	279	0.302	833	1.39	
2600	0.424	2357	2663	77.6	83.5	409	0.776	$4.17.10^{-9}$	1.405	
2700	0.422	2437	2770	81.2	100.5	598	1.74	20.4	1.42	
2800	0.420	2517	2878	84.8	119.0	823	3.74	83.3	1.43	
2900	0.417	2596	2986	88.5	139	1110	7.57	309		
3000	0.415	2675	3094	92.3	162	1490	14.9	$1.05.10^{-6}$		
3100	0.413	2753	3202	97.0	189	1960	28.1	3.31		
3200	0.411	2829	3311	99.9	221	2530	50.5	10.0		
3300	0.409	2903	3422	103.8	254	3250	87.7	26.3		
3400	0.407	2978	3533	107.8	291	4080	149	70.8		
degr. K		degr. K	degr. K	$\mu\Omega$ cm	Watts cm^{-2}	I.C. cm^{-2}	Amp. cm^{-2}	gr. cm^{-2} sec.$^{-1}$	Watts cm^{-1} degr.$^{-1}$	μV.degr.$^{-1}$

For the purpose of measuring the *thermionic emission* lamps were constructed in which the filament was surrounded by an electrode in the usual manner. A positive charge given to this electrode collected the emitted electrons. The electron emission is given by the formula:

$$i = AT^2 e^{-\frac{b}{T}} \ \mathrm{Amp./_{cm^2}}$$

in which
$$A = 60{,}2$$
$$b = 52230.$$

The rate of vaporization has been determined from the decrease in diameter of a glowing tungsten filament maintained at a constant temperature, as calculated from the increase of its resistance, measured as a function of time. For the rate of vaporization the following formula holds:

$$\log m = 11{,}92 - \frac{4.84.10^4}{T} - 0{,}368 \log T - 0{,}00016\, T.$$

From the observed temperature variation near the leads, *heat conductivity* determinations were made. The positive and the negative filament end showed a somewhat different temperature variation. This is caused by the *Thomson effect*; the Thomson coefficient can be calculated from the ratio of the temperature gradients at both leads.

The *colour temperature* of tungsten at a temperature T is defined as

TABLE II. The candle power of the black body

T	B
1300	0.015
1336	0.111
1400	0.261
1500	0.818
1600	2.26
1700	5.74
1800	12.4
1900	25.7
2000	50.1
2100	91.6
2200	156
2300	256
2400	410
2500	620
2600	915
°K	I.C./cm^2

the temperature of the black body to obtain the same energy distribution in the visible spectrum as tungsten gives. The colour temperature is of great importance because of the possibility to calculate the candle power of the black body from the relation between brightness temperature, colour temperature and candle power of tungsten by the formula:

$$B_{\text{Black Body at } T_c} = B_{\text{Tungsten at } S} \cdot e^{\frac{C_2}{2{,}303\,\lambda}\left(\frac{1}{S}-\frac{1}{T_c}\right)}.$$

Furthermore direct candle power determinations have been done of the black body. In this study the pyrometer bench was used as a microphotometer, it being standardized for white as well as for red (monochromatic) light. The red and the white brightness of the black body were measured in immediate succession, the red brightness determining the temperature. These measurements extended over the temperature range from 1300 to 2600° K, and were in complete concordance with the brightness measurements of tungsten filaments. Our data on the candle power of the black body are given in table II.

These measurements give for the least mechanical equivalent of light the value:

$$M = 0{,}00146 \text{ Watts per Intern. Lumen.}$$

Eindhoven, May 11, 1925. *Natuurkundig Laboratorium der*
N.V. PHILIPS' *Gloeilampenfabrieken.*

Mathematics. — "*Intuitionistischer Beweis des Jordanschen Kurvensatzes.*" By Prof. L. E. J. Brouwer.

(Communicated at the meeting of June 27, 1925.)

§ 1. Unter einer *ebenen Kernspezies* verstehen wir eine Spezies von Punktkernen der Ebene.

Unter einer *geschlossenen stetigen Kurve* S verstehen wir eine ebene Kernspezies, welche eindeutiges Bild eines mit einem Umlaufssinne versehenen (als Kernspezies zu betrachtenden) Quadratumfanges K ist.

Nach Bd. XXVII, S. 193 dieser Proceedings ist die geschlossene stetige Kurve S *gleichmässig stetiges* Bild des Quadratumfanges K.

Unter einer *Jordanschen Kurve* J verstehen wir eine geschlossene stetige Kurve, welche als Abbildung des Quadratumfanges K eine stetige (mithin eindeutige und nach Bd. XIII, Nr. 2, S. 4 der Verhandelingen dieser Akademie auch gleichmässig stetige) Umkehrung besitzt.

Auf Grund der Beziehung zwischen J und K existiert zu jedem ε ein solches mit ε gegen 0 konvergierendes (d.h. positiv-konvergierendes) ε_1, dass die Bildpunktkerne A' und B' auf K je zweier Punktkerne A und B von J, deren Entfernung $< \varepsilon$ 1) ist, eine Entfernung $< \varepsilon_1$ besitzen, so dass auch die Breite von einem der Bogen $A'B'$ von K kleiner als ε_1 ist und somit die Breite von einem der Bogen AB von J kleiner als ε_2, wo ε_2 mit ε_1, also mit ε, gegen 0 konvergiert.

Sei nun ε so gewählt, dass das zugehörige $\varepsilon_2 < \dfrac{1}{3}$ der Breite von J ist, und seien A, B und C drei derartige Punktkerne von J, dass $\varrho(A, C) < \frac{1}{2}\varepsilon$ und $\varrho(C, B) < \frac{1}{2}\varepsilon$. Alsdann kann man entweder einen A, B und C enthaltenden Bogen von J der Breite $< \varepsilon_2$ oder für je zwei der Punktkerne A, B und C einen Abstand > 0 bestimmen. Im letzteren Falle sei $A\alpha B$ ein Bogen von J der Breite $< \varepsilon_2$; zu diesem Bogen muss C entweder gehören, oder von ihm einen Abstand > 0 besitzen. Im ersteren Falle hat man wiederum einen A, B und C enthaltenden Bogen von J der Breite $< \varepsilon_2$; im letzteren Falle unterscheiden wir zwei Unterfälle: 1. Es gibt einen den Punktkern B enthaltenden Bogen $A\alpha\, B\beta\, C$ von J der Breite $< \varepsilon_2$, der also wiederum als A, B und C enthaltender Bogen von J der Breite $< \varepsilon_2$ auftritt. 2. Es gibt einen Bogen $C\gamma\, A$ von J der Breite $< \varepsilon_2$, von welchem B eine Entfernung > 0 besitzt. Alsdann hat derjenige Bogen $C\,B$ von J, von dem A eine Entfernung > 0 besitzt, eine Breite $> \varepsilon_2$, so dass die Breite des Bogens $C\gamma\, A\alpha\, B$ von J kleiner als ε_2 sein muss,

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1) Die Formel $a > b$ (*a grösser als b*), wo a und b reelle Grössen vorstellen, besagt in diesem Aufsatze, dass eine natürliche Zahl n angegeben werden kann, so dass $a - b \geq 2^{-n}$.

und wiederum ein A, B und C enthaltender Bogen von J der Breite $< \varepsilon_2$ existiert. Aus der Annahme dass $\varrho(A, C) < \frac{1}{2}\varepsilon$ und $\varrho(C, B) < \frac{1}{2}\varepsilon$ folgt mithin für drei Punktkerne A, B und C von J *immer* die Existenz eines A, B und C enthaltenden Bogens von J der Breite $< \varepsilon_2$.

Sei nun in J eine $\frac{1}{2}\varepsilon$-Kette gegeben. Wenn wir je zwei aufeinanderfolgende Punkte dieser Kette durch einen Bogen von J der Breite $< \varepsilon_2$ verbinden, erzeugen wir eine ganze (positive, negative oder verschwindende) Totalanzahl von Umläufen von J, die wir die *Ordnung* der Kette nennen. Auf Grund des Resultates des vorstehenden Absatzes *wird die Ordnung einer in J gelegenen $\frac{1}{2}\varepsilon$-Kette durch eine $\frac{1}{2}\varepsilon$-Abänderung in J nicht geändert.*

§ 2. Sei P ein von J *vollständig entfernter*, d.h. von J einen Abstand > 0 besitzender Punkt. Wir wählen $\varepsilon'' < \frac{1}{2}\varepsilon$ in solcher Weise, dass $\varrho(P, J) > \varepsilon'' > 0$. Sei ε' eine solche Grösse $< \frac{1}{2}\varepsilon''$, dass für je zwei in einer Entfernung $< \varepsilon'$ voneinander gelegene Punktkerne A und B von J die Breite von einem der Bogen AB von $J < \varepsilon''$ ist. Sei weiter $\varkappa^0$ eine in J gelegene *kanonische*, d.h. J genau einmal im positiven Sinne ohne Umkehrung des Richtungssinnes durchlaufende ε'-Kette und $\varkappa_2$ eine beliebige in J gelegene ε'-Kette. Alsdann kann $\varkappa_2$ in J durch eine endliche Folge von ε''-Abänderungen *in bezug auf $\varkappa^0$ kanonisiert*, d.h. in eine solche Kette $\varkappa_3$ übergeführt werden, welche aus einer ganzen (positiven, negativen oder verschwindenden), mit der Ordnung von $\varkappa_2$ identischen Anzahl von aufeinanderfolgenden mit $\varkappa^0$ zusammenfallenden Ketten besteht.

Sei Q ein Quadrat, das in seinem Inneren P und in einer Entfernung $> \varepsilon''$ von seinem Umfang J enthält. Wir zerlegen Q in kongruente, homothetische Teilquadrate q mit einer Seitenlänge $< \frac{1}{16}\varepsilon'$. Unter diesen q-Quadraten wählen wir eine Teilspezies τ'' aus, welche alle in einer Entfernung $> \frac{1}{4}\varepsilon'$ von J liegenden q-Quadrate enthält, während alle zu ihr gehörenden q-Quadrate in einer Entfernung $> \frac{1}{8}\varepsilon'$ von J gelegen sind. Der Teilspezies τ' fügen wir alle diejenigen q-Quadrate hinzu, welche in einem von τ' bestimmten, J nicht enthaltenden Komplementärbereich gelegen sind, und demzufolge einen Abstand $> \frac{1}{8}\varepsilon'$ von J besitzen. Durch diese Hinzufügung geht τ' in eine Spezies τ von q-Quadraten über, welche innerhalb Q nur einen einzigen Komplementärbereich bestimmt und alle in einer Entfernung $> \frac{1}{4}\varepsilon'$ von J liegenden q-Quadrate enthält, während alle zu ihr gehörenden q-Quadrate in einer Entfernung $> \frac{1}{8}\varepsilon'$ von J gelegen sind.

Sei β^0 der grösste P enthaltende ganz von τ überdeckte Bereich. Der nicht im Umfange von Q gelegene Teil der Grenze von β^0 (ebenso wie von einem beliebigen unausdehnbaren ganz von τ überdeckten Bereich) ist zusammenhängend und jeder seiner Punkte besitzt von J eine Entfernung $< \frac{3}{8}\varepsilon'$ und $> \frac{1}{8}\varepsilon'$. Wir unterscheiden nun zwei Fälle:

1. Der Umfang von Q gehört nicht zur Grenze von β^0. Sei $\varkappa'$ eine aus in der Grenze π von β^0 in positivem Sinne in jeweiligen

Abständen gleich $\frac{1}{16}\varepsilon'$ aufeinanderfolgenden Eckpunkten von q-Quadraten bestehende, in π kanonische Kette. In bezug auf diese Kette $\varkappa'$ (d.h. in bezug auf die aus den Verbindungsstrecken der sukzessiven Punkte von $\varkappa'$ hervorgehende geschlossene polygonale Linie) besitzt P die Ordnung 1. Indem wir der Reihe nach jeden Punkt von $\varkappa'$ durch einen in einer Entfernung $< \frac{3}{8}\varepsilon'$ gelegenen Punkt von J ersetzen, erzeugen wir eine Reihe von Ketten $\varkappa_1', \varkappa_2', \varkappa_3', \ldots \varkappa_m'$, wo $\varkappa_m' = \varkappa''$ eine in J gelegene ε'-Kette vorstellt. Sei $\varkappa_p' = \ldots RST \ldots$ und $\varkappa_{p+1}' = \ldots RS_1T \ldots$, dann kann weder das Dreieck RSS_1 noch das Dreieck TSS_1 den Punkt P in seinem Innern enthalten. Hieraus folgt, dass P in bezug auf $\varkappa_p'$ und $\varkappa_{p+1}'$ für jedes p die gleiche Ordnung, mithin auch in bezug auf $\varkappa'$ und $\varkappa''$ die gleiche Ordnung besitzt, so dass die Ordnung von P in bezug auf $\varkappa''$ gleich 1 ist.

Sei $\varkappa_g' = \ldots RSU \ldots$ bzw. $\ldots RU \ldots$ eine in J gelegene ε''-Kette, welche durch eine ε''-Abänderung in eine in J gelegene ε''-Kette $\varkappa_h' = \ldots RTU \ldots$ übergeht. Alsdann ist es ausgeschlossen, dass eines von den Dreiecken RST und UST bzw. dass das Dreieck RTU den Punkt P in seinem Innern enthält, so dass P in bezug auf $\varkappa_g'$ und in bezug auf $\varkappa_h'$ die gleiche Ordnung besitzt. Wenn wir also die im vorigen Absatz bestimmte Kette $\varkappa''$ durch eine endliche Folge vón ε''-Abänderungen in J in bezug auf $\varkappa^0$ kanonisieren, erhalten wir eine in J gelegene ε'-Kette $\varkappa$, in bezug auf welche P noch immer die Ordnung 1 besitzt.

Wenn wir je zwei aufeinanderfolgende Punkte von $\varkappa$ durch einen Bogen von J der Breite $< \varepsilon''$ verbinden, erzeugen wir eine eine ganze Totalanzahl a von Umläufen von J darstellende geschlossene stetige Kurve V, in bezug auf welche P die Ordnung 1 besitzt, so dass a nicht verschwinden kann und die Ordnung von P in bezug auf J gleich $\frac{1}{a}$ ist. Weil aber die letztere Ordnung eine ganze Zahl sein muss, so ist a entweder gleich $+1$ oder gleich -1, und auch *die Ordnung von P in bezug auf J entweder gleich $+1$ oder gleich -1.*

2. Der Umfang von Q gehört zur Grenze von β^0. Alsdann kann P durch eine in einer Entfernung >0 von J gelegene unendliche polygonale Linie mit dem Unendlichen verbunden werden, so dass *die Ordnung von P in bezug auf J gleich 0 ist.*

Aus den für die obigen Fälle 1. und 2. hergeleiteten Ordnungseigenschaften folgt, dass für einen beliebigen von J vollständig entfernten Punkt P die Entscheidung zwischen den Fällen 1. und 2. unabhängig von der Wahl der (den Bedingungen $\varepsilon'' > 0$, $\varepsilon'' < \frac{1}{2}\varepsilon$ und $\varepsilon'' < \varrho(P, J)$ genügenden) Grösse ε'' ausfällt.

§ 3. Seien P_1 und P_2 zwei von J vollständig entfernte Punkte. Wir unterscheiden drei Fälle:

1. P_1 und P_2 befinden sich beide im ersten Falle von § 2. Alsdann wählen wir ε'' in sowohl für P_1 wie für P_2 passender Weise,

d.h. so dass sowohl P_1 wie P_2 eine Entfernung $> \varepsilon''$ von J besitzt, und bestimmen nach § 2 ein entsprechendes Quadrat Q, eine entsprechende Spezies τ von q-Quadraten, und in τ die zu P_1 bzw. P_2 gehörigen Bereiche $\beta_1{}^0$ und $\beta_2{}^0$. Wären nun $\beta_1{}^0$ und $\beta_2{}^0$ verschieden, dann besässe P_2 in bezug auf $\varkappa'_1$ die Ordnung 0, mithin der Reihe nach auch in bezug auf $\varkappa_1''$, $\varkappa_1$, V_1 und J die Ordnung 0. Aus diesem Widerspruche folgern wir, dass $\beta_1{}^0$ und $\beta_2{}^0$ identisch sind, *dass also P_1 und P_2 durch einen von J vollständig entfernten endlichen Streckenzug verbunden werden können.*

2. P_1 und P_2 befinden sich beide im zweiten Falle von § 2. Alsdann wählen wir wiederum ε'' in sowohl für P_1 wie für P_2 passender Weise, und bestimmen nach § 2 ein entsprechendes Quadrat Q und eine entsprechende Spezies τ von q-Quadraten. Hierauf kann sowohl P_1 wie P_2 durch einen von J vollständig entfernten endlichen Streckenzug mit dem Umfang von Q verbunden werden, *so dass P_1 und P_2 auch untereinander durch einen von J vollständig entfernten endlichen Streckenzug verbunden werden können.*

3. Von den Punkten P_1 und P_2 befindet sich der eine im ersten, der andere in zweiten Falle von § 2. *Alsdann ist die Existenz eines von J vollständig entfernten, P_1 und P_2 verbindenden Streckenzugs ungereimt,* weil auf einem derartigen Streckenzuge einerseits alle Punkte gleiche, andererseits die Punkte P_1 und P_2 verschiedene Ordnungen in bezug auf J haben müssten.

Diejenigen von J vollständig entfernten Punkte, die sich im ersten bzw. im zweiten Falle von § 2 befinden, nennen wir *positiv-innere* bzw. *positiv-äussere* Punkte von J.

§ 4. Positiv-äussere Punkte von J lassen sich in mannigfacher Weise unmittelbar angeben. Einen positiv-inneren Punkt von J bestimmen wir wie folgt:

Sei ε der in § 1 formulierten Forderung entsprechend gewählt, und ε'' eine der Beziehung $\frac{1}{32}\varepsilon > \varepsilon'' > 0$ genügende Grösse. Sei ε' im Anschluss an ε'' so gewählt, dass der im ersten Absatz von § 2 angegebenen Beziehung zwischen ε' und ε'' genügt wird. Sei Q ein Quadrat, das in einer Entfernung $> \varepsilon''$ von seinem Umfang J in seinem Inneren enthält. Zu Q und ε' konstruieren wir nach der im zweiten Absatz von § 2 angegebenen Methode eine entsprechende Spezies von q-Quadraten τ. Sei β_1 der grösste an den Umfang von Q grenzende, ganz von τ überdeckte Bereich. Dieser Bereich β_1 bestimmt innerhalb Q einen einzigen Komplementärbereich K, der J enthält und dessen Grenze zusammenhängend ist. *Wir wollen einen Augenblick annehmen, dass jedes in einem sowohl zu K wie zu τ gehörigen Bereich enthaltene, aus q-Quadraten bestehende Quadrat eine Seitenlänge $< \frac{1}{16}\varepsilon$ besitzt.*

Alsdann hat jeder Punkt von K eine Entfernung $< \frac{3}{8}\varepsilon' + \frac{1}{8}\varepsilon$ von J. Hieraus folgern wir, dass die im ersten Absatz von § 2 definierte kanonische ε'-Kette $\varkappa^0$ sich mittels einer endlichen Folge von zunächst ε''-Abänderungen und sodann $(\frac{3}{4}\varepsilon' + \frac{1}{4}\varepsilon)$-Abänderungen in J, mithin mittels einer endlichen Folge von $\frac{1}{2}\varepsilon$-Abänderungen in J, in eine aus einem einzigen Punkt von J bestehende singuläre Kette überführen lässt. Dies aber widerspricht der am Schluss von § 1 formulierten Eigenschaft. *Es gibt also einen solchen unausdehnbaren, ganz von τ überdeckten, nicht an den Umfang von Q grenzenden Bereich β, in welchem ein aus q-Quadraten bestehendes Quadrat g der Seitenlänge $> \frac{1}{16}\varepsilon - \frac{1}{4}\varepsilon'$ enthalten ist.*

Ein zum Mittelpunktkerne von g gehöriger Punkt P genügt der Beziehung $\varrho\,(P, J) > \frac{1}{32}\varepsilon$, so dass für die Entitäten P, ε'' und ε' dieses § alle für die im § 2 eingeführten Entitäten P, ε'' und ε' daselbst geforderten Beziehungen bestehen. Auf Grund der ersten drei Absätze des § 2 *erscheint mithin der Bereich β als grösster P enthaltender, ganz von τ überdeckter Bereich β^0, und befindet sich als solcher im ersten der beiden dortigen Fälle,* d. h. wir erkennen in P einen positiv-inneren Punkt von J.

§ 5. Unter einem *stetigen Kurvenbogen* verstehen wir eine ebene Kernspezies, welche eindeutiges, mithin gleichmässig stetiges Bild des (als Kernspezies zu betrachtenden) Einheitsintervalles der X-Achse ist.

Unter einem *Jordanschen Kurvenbogen* F verstehen wir einen stetigen Kurvenbogen, der als Abbildung des Einheitsintervalles der X-Achse eine stetige (mithin eindeutige und gleichmässig stetige) Umkehrung besitzt.

Sei P ein von F vollständig entfernter Punkt. Wir wählen ε'' in solcher Weise, das $\varrho\,(P, F) > \varepsilon'' > 0$. Sei ε' eine solche Grösse $< \frac{1}{2}\varepsilon''$, dass für je zwei in einer Entfernung $< \varepsilon'$ voneinander gelegene Punktkerne A und B von F die Breite des Bogens AB von $F < \varepsilon''$ ist. Alsdann kann eine beliebige in F gelegene ε'-Kette in F durch eine endliche Folge von ε''-Abänderungen *zusammengezogen*, d.h. in eine aus einem einzigen Punkte bestehende "singuläre Kette" übergeführt werden.

Sei Q ein Quadrat, das in einer Entfernung $> \varepsilon''$ von seinem Umfang F in seinem Innern enthält. In analoger Weise wie im § 2 konstruieren wir in Q die q-Quadrate, und bilden eine Spezies τ von q-Quadraten, welche innerhalb Q nur einen einzigen Komplementärbereich bestimmt und alle in einer Entfernung $> \frac{1}{4}\varepsilon'$ von F liegenden q-Quadrate enthält, während alle zu ihr gehörenden q-Quadrate in einer Entfernung $> \frac{1}{8}\varepsilon'$ von F gelegen sind. Sei β^0 der grösste P enthaltende ganz von τ überdeckte Bereich. Nehmen wir an, dass der Umfang von Q nicht zur Grenze von β^0 gehört, und erörtern wir diese Voraussetzung analog wie im § 2. Alsdann gelangen wir zu einer in F gelegenen ε'-Kette $\varkappa''$, in bezug auf welche P die Ordnung 1 besitzt. Wenn wir aber durch eine endliche Folge von ε''-Abänderungen diese Kette $\varkappa''$ in eine aus einem einzigen Punkte bestehende singuläre Kette $\varkappa$ überführen, dann muss die Ordnung von P in bezug auf $\varkappa$ einerseits gleich 1, andererseits gleich 0 sein. Aus diesem Widerspruche ergibt sich, dass *der Umfang*

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von Q *notwendig zur Grenze von* β^0 *gehören muss*, und hieraus nach der Methode des § 3 weiter, dass *je zwei von* F *vollständig entfernte Punkte durch einen von* F *vollständig entfernten Streckenzug verbunden werden können.*

Wenn wir jetzt zur Jordanschen Kurve J zurückkehren, so folgt aus dem vorstehenden Absatze, dass für einen beliebig kleinen Teilbogen β von J ein beliebiger positiv-innerer und ein beliebiger positiv-äusserer Punkt von J durch einen vom Komplementärbogen von β in J vollständig entfernten endlichen Streckenzug verbunden werden können. Mittels einer kurzen Ueberlegung folgt hieraus weiter, dass zu einem beliebigen Punktkern A von J sowohl eine gegen A positiv konvergierende Folge von positiv-inneren Punkten von J wie eine gegen A positiv konvergierende Folge von positiv-äusseren Punkten von J konstruiert werden kann, m. a. W. dass *jeder beliebige zu* J *gehörige Punkt sowohl Grenzpunkt von positiv-inneren wie Grenzpunkt von positiv-äusseren Punkten von* J *ist.*

Sei G_1 die Spezies der positiv-inneren, G_2 die Spezies der positiv-äusseren Punkte von J. Alsdann muss jeder Grenzpunkt von G_1 abweichen von G_2 und jeder Grenzpunkt von G_2 abweichen von G_1. Sei γ_1 die *Grenze von* G_1, d.h. die Spezies der von G_1 abweichenden Grenzpunktkerne von G_1 und γ_2 die *Grenze von* G_2, d.h. die Spezies der von G_2 abweichenden Grenzpunktkerne von G_2. Auf Grund des Resultates des vorigen Absatzes gehört dann jeder Punktkern von J sowohl zu γ_1 wie zu γ_2. Andererseits folgt aus der am Anfang des laufenden Absatzes befindlichen Aussage, dass jeder Punktkern von γ_1 sowie jeder Punktkern von γ_2 einen Abstand Null von J besitzen, mithin zu J gehören muss.

Somit ist sowohl von der Spezies der positiv-inneren wie von der Spezies der positiv-äusseren Punkte von J *die Grenze mit* J *identisch.*

Hystology. — *"On collagenous fibrils; their origin, structure and arrangement."* By G. C. HERINGA and H. A. LOHR. (Communicated by Prof. J. BOEKE.)

(Communicated at the meeting of Oct. 25. 1924).

The paraffin-, and celloidin-technique has in course of time come to be looked upon as an infaillible method, so that implicit confidence is placed in the histological images thereby obtained. Sometimes, however, their reliability must be called in question, when we see that the various current hypotheses concerning the structure of connective tissue have been based on observations of material that suffered considerably from the deleterious influence of the embedding method. Two factors, viz dehydration and heating are answerable for this injury. Their intense influence may be readily conceived when comparing a carefully made paraffin-preparation with one that has been embedded in gelatin and cut on the freezing-microtome. Then it will appear that the injury done to the structure of the connective tissue by the said influences, is greater than in other more cellulous tissues. This injury increases with the augmentation of collagenous fibrils in the tissue. With justice we may here denounce the incriminated methods on account of the difficulty to obtain with them good preparations of organs such as tendon and skin, which have almost entirely been made up of highly fibrillous connective tissue. In the paraffin-sections of the skin the fibrilbundles are shrivelled up into shapeless lumps leaving wide interstices. In consequence of the shrinking a tendon of some thickness, is apt to grow so hard, that cutting it is altogether out of the question. The cells of the connective tissue are also involved in the general ruin. During the shrinking of the fibrils, surrounding them, they are torn asunder to all sides. What remains of them is only a number of scattered pieces and fragments, only the nuclei can be discerned in a hematoxylin-eosin preparation amidst the red-stained chaos. This aspect has become so familiar to authors that they have gradually forgotten, that the technique adopted destroyed the tissue and they have based on their observations of mutilated cell-rests the conclusion that fibroblasts, after accomplishing their task of producing fibrils, are "physiologically degenerated". This conclusion is absolutely wrong! True, unlike the fibrillar connective tissue, the young wide-meshed connective tissue presents with the same paraffin-technique beautiful branched cells, and these cells screen themselves from observation according as the fibrils accumulate around them. (FLEMMING's images). For this fact, however, an equally plausible explanation may be found, when considering that the fibril-poor tissue is far less sensitive to the hurtful influence of the method than the collagenous fibrils. This is the only possible conclusion when we consider that in gelatin-sections, where shrinking from dehydration does not take place, in the collagenous fibres, no "degenerative" modifications of the connective tissue-cells are at all noticeable, and that, on the other hand, also in "adult" connective tissue the cells appear as perfectly vital protoplasma-rich elements, which are just as much as their mesenchymal progenitors interconnected on all sides by their branches. In previous publications [1]) one of us has discussed the cells of connective tissue in extenso. We should not have mentioned it here, if not, as stated just now, some authors are inclined to associate that presumed cell-evolution with a "fibroblastic" function of the cell.

In the gelatin-freezing-sections of the skin the interspaces between the cells are so of collagenous fibres, that at best only inappreciably small capillary interstices are available for tissue-lymph. Instead of the chaos of the paraffin-sections we find here well-defined bundles of fibres, known also from the "artificial oedema preparations", which form as yet an inextricable closely packed network. The spatial orientation of the cells inter se and their branches on the one side, and on the other that of the intertwining collagenous bundles between the cells, is so complicate that it is simply impossible to establish a directly genetical connection between cells and fibres. This conviction, which urged itself upon us the moment we took cognizance of gelatin-sections of connective tissue, induced us to try to prove, with our technique, the old hypothesis of NÄGELI, who maintained that the fibres appear spontaneously in the interstitial substance. Our gelatin technique gave us an advantage over the authors who also tried before us to abandon the "fibroblastic" theory (V. EBNER, MERKEL and others) while, moreover, we found a powerful aid in the application of dark-ground microscopy. While our embedding method guarded the fibres from shrivelling and clotting, dark-ground microscopy enabled us to demonstrate the formation of extremely fine (ultramicroscopic) collagenous elements in regions where not a trace of them could be recognized with the ordinary illumination; SIEDENTOPF's "Wechsel-condensor" did inestimable service. Until some weeks ago we were ignorant of the existence of SZEGVARI's "Azimuthblende". Therefore, when trying to recognize the fibrils on the basis of SIEDENTOPF's [2]) and KRUYT's [3]) researches, we made use of a unilateral illumination by inserting a slit-screen. Especially when, moreover, the peripheral rays are intercepted with the iris, the results are usable, though we must admit that the adjustable "Azimuthblende" of SZEGVARI is far preferable. Be this as it may, we have attained also with our somewhat primitive instruments a few results that will be reported below. Our material consisted in umbilical cords and skin of sheep and man of various life-periods.

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[1]) HERINGA, Ned. Tijdschr. v. Geneesk. 1922, '23 and '24.

[2]) SIEDENTOPF, Z. W. Mikrosk. Bd. 29, 1912.

[3]) KRUYT, These Proceedings **24**, 1664, 1916.

1. (umbilical cord sheep's embryo 2.3 cm). When collagenous fibrils appeared, we signalled in an initial stage the appearance of diffuse Tyndal-light in the interstitial substance, in places where neither with common illumination, nor with dark-ground illumination fibrils could be distinguished. They were visible, however, in the neighbourhood of fibrillar formations. Rotation of the slit screen revealed the presence of needle-shaped ultramicrones.

2. Already in the youngest of the available umbilical cords, the same we referred to above, we found fibrils anywhere in the interstitial substance.

Large masses of fibrils, coming directly into view with light-ground, are disposed round the cells and their prolongations like inextricable clews, leaving the cells in their midst intact. Fine-spun fibrils, only visible in dark-ground are everywhere seen isolated or in groups, also at a large distance from the cells. Individually they proceed in a strongly curving line, so that, when they are proceeding in groups, they are intertwined in a very complicated way.

3. In many places in the youngest umbilical cord, and everywhere in the older ones a tendency to orderly arrangement in the fibril-complexes may be noted resulting in the formation of bundles of fibrils running in the same direction. These bundles are never cylindrical. From their very origin to the stage of the fully developed connective tissue they are tape-shaped, i.e. the fibrils do not range themselves evenly on all sides, not cylindrically, but exclusively or at any rate by preference, in a plane, the one beside the other. Just like the individual fibrils above-discussed, the course of the tapes is complicate and twisted. This is very likely due to the ultrastructure of the fibrils themselves, because, as has been stated before, the morphological character of the bundles is so irregular as to preclude any responsibility of the individual cells in this respect, even though the most strongly developed fibrillar formations are always found in the neighbourhood of the cells. We detected in our preparations MERKEL's images of fibril-masses in the environment of the cells. On closer inspection these fibrilmasses always appeared to consist of intertwined and anastomotic fabril-tapes, mostly gathered into films.

4. In contradistinction to the irregularity in the course of the individual collagenous fibrils and their tapes (in umbilical cord and skin), the regularity with which the tapes are united into one whole of three dimensions implies a compliance with certain laws, which we have not yet been able to analyse, but which undoubtedly points to a three-dimensional architecture in the differentiation of the interstitial substance as well as in the disposition of the cells ¹).

5. To sum up we arrive at the provisional conclusion that the collage-

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¹) HERINGA, l. c.

nous fibrils in the interstitial substance originate from needle-shaped ultramicrones ranging themselves in area extent; secondly that the interstitial substance does not induce the formation of fibrils everywhere in an equal degree; further that ceteris paribus the environment of the cells are selected by preference for the distribution of the fibril-masses; also that cells exert no influence upon the course of the fibrils; and finally that the tendency of fibrils to a peculiarly sinuous course, as well as their habit to form close tapelike complexes must arise from physical properties of the fibrils themselves.

We do not disguise from ourselves, though, that the above conclusions are extremely fragmentary, and that our present achievement is but the first step on an endless field of research. Meanwhile taking into consideration that the results of our work of nearly two years, inadequate though they may be, seem to be fairly complementary to the researches made by others in a similar domain, we feel justified in publishing our view as a basis for subsequent investigation.

As stated before, our working hypothesis was the well-known hypothesis of NÄGELI who imagined the collagenous fibrils to be rows of micellae.

Already many years ago this ingenious hypothesis was substantiated by the inquiries of AMBROSE and his pupils. It was, moreover, corroborated pre-eminently in the röntgenographic researches of HERZOG, JANCKE and others, who on the principle suggested by LAUE, established for good and all the micellar formation of a number of organic fibrillar structures and also of the collagenous fibrils. Some months ago we also started, apart from HERZOG, a röntgenological investigation of the tendon; we were benefited in our endeavours by Dr. KOLKMEIJER's knowledge and his instrument which he kindly placed at our disposal. For the present this investigation seems to confirm HERZOG's results, so that we are encouraged to pursue it.

If we consider how, when seen in the light thrown upon them by the röntgenological investigations alluded to, the fibril assumes completely the character of a crystalline formation, we cannot be surprised that this induced several authors to call the origin of fibrils a crystallization-process, their main point of consideration then being that it must be in virtue of intermicellar forces that the ultramicrones "eine gesetzmäszige Anordnung einnehmen" [1]). The more we look about us in living nature, the more we are impressed with the idea that the orderly arrangement of the micellae is one of the chief resources of nature to attain the formation of sharply differentiated structural elements. It is especially the fibrils with their row of micellae that come into play in a great number of modifications in living nature.

So while of late years the hypothesis that collagenous fibrils consist of ultramicrones arranged in a row, has gained ground more and more, in histology the conception of the origin of connective-tissue fibres remains circumscribed within the field of old speculations. Our inquiry of which the results have been recorded above, is intended for an effort to assign to physical data the influence on histology that is properly due to them.

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[1]) SCHMIDT. „Die Bausteine des Tierkörpers im polarisierten Lichte". 1924.

While on the one hand we succeeded in identifying the origin of the collagenous fibrils as a "crystallization" process that progresses spontaneously in a given environment, on the other hand their arrangement in the interstitial substance remains a puzzle. We know, indeed, for certain from the analytical work of ROUX c.s., and from the experimental work of LERICHE, that external mechanical factors exert a great influence upon the disposition of fibrils; however, when examining the complex fibril-texture of, say, the skin, we see at a glance that, besides external factors also intrinsic forces come into play. We here wish to emphasize one of the peculiar behaviours of collagenous fibrils, viz. the formation of bundles. It has been known long since that collagenous fibrils display a tendency in their joint-course to combine into bundles. This has also raised the supposition of splitting of fibrils. It seems strange, however, that, so far as I know, workers have never thoroughly considered the fact that the bundles are not round but always flat like a tape. Still, these details have often been distinctly represented in many pictures, e.g. in V. EENER's and LAGUESSE's publications. As regards the forces that keep these fibrilbundles together, histologists speak of a sticky substance encasing the fibrils. In the recently published work by SCHMIDT (l. c.), which for the rest is so much up-to-date, this term is met with again and again.

We have suspected that the peculiar junction of the fibrils, just as well as that of the micellae, may take origin in the specific properties of the fibrils themselves. We are the more disposed to think so since SZEGVARI [1]) found in various threadforming colloids a number of unidirectional bundles of filiform elements. In our endeavour to find an analogy between the observations of SZEGVARI and ours we are prompted to consider more narrowly the collagenous fibrils and their origin, from a collochemical point of view.

It need not be argued that gelatinous fibril-poor interstitional substance, such as is typical of embryonic connective tissue, is of a colloidal nature. It is also a fact, however, that the collagenous fibres, which we see appear increasingly in this interstitial substance, also reveal the typical properties of colloids. The sensitivity to dehydrating means, referred to at the outset, has justly convinced NÄGELI that the micellae, which as stated above, make up the fibrils, must be enclosed in water. The current opinion now is, that a water-envelop belongs to all protein bodies in colloidal solution, (emulsoids).

Now, when concluding on the basis of the arguments enumerated above, that these fibrils have not been introduced into the intercellular substance

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1) SZEGVARI. Ueber die ultra mikroskopische Untersuchung linearer Elemente. I u. II Zeitschr. f. phys. Chemie, 112. (1924).

from the outside (i.e. from the cells) but have originated there spontaneously through methodical conjunction of isolated particles; moreover, when reflecting that, as fibrillation progresses, the consistency of the connective tissue increases and its water-content decreases in the course of the histogenesis ; when all these facts are considered collectively, then it is conceivable that we should designate the change, accomplished in the interstitial substance as a gelification. We are the more entitled to do so as in collochemistry gel is defined as a precipitation of hydrated particles, which definition is quite applicable to the process under consideration. A peculiar feature of this case, however, is that the particles close up more firmly in one spatial direction than in the other two. From the above observations regarding a systematic grouping of the fibrils, it may be inferred that also in the other two directions there is decidedly a system in the orientation of the particles. This, in our opinion, is an additional argument for defining connective-tissue as a thread-forming gel.

In colloid chemistry we know a number of substances in which the precipitation of particles also gives rise to filiform fibrillation (soap, vanadium pentoxid). We do not quite see why SZEGVARI, contrary to the current opinion, speaks of threadforming sols, and not of threadforming gels. SZEGVARI himself detected, as we observed just now, in these "sols" a constancy of every individual group in the orientation of the threads *inter* se. If this formation of bundles is of more frequent occurrence in our preparations than in those of SZEGVARI, which has convinced us of the existence of a systematic three-dimensional structure, we cannot help observing at the same time, that the histological technique, which has been so much disputed, offers the advantage of affording some insight into spatial structures, which are of necessity lost in the preparations of the colloid-chemists. This very consideration prompted us to publish our views after reading SZEGVARI's publications. In view of the grounds for considering connective tissue as a thread-forming colloid, it would seem to us that our preparations are apt to render SZEGVARI's observations more conspicuous.

Finally we intend to set forth hypothetically how we would imagine the structure of collagenous fibrils from a colloidchemical point of view. It would appear to us, as we shall endeavour to prove, that this speculation affords a satisfactory explanation of a number of well-known physical properties of collagenous subsance.

Of course we started again from NÄGELI's hypothesis concerning the formation of fibrils from a linear succession of particles. The result obtained with polarized light (AMBRONN, SCHMIDT and others) together with our observations with dark-ground illumination (vide supra) warrant the assumption, that just as in the case of threadgels (SZEGVARI) and fibrin (HEKMA), also in the case of collagenous fibres the micellae are rod-shaped and ellipsoidal. The micellae, being emulsoids, are further possessed of two stabilityfactors, viz. 1° a micellar electrical charge and 2° a water-envelop.

Finally we also feel inclined to assume that the elongated micellae have beyond their micellar charge also a polar charge of opposite sign at their extremities, which charge we hold to be answerable for the arrangement of particles in rows. The size of the particles, the distance of which will, as we hope, be elucidated by our röntgenphoto's, we suppose to be amicronic. Supposing the water-envelop to be an unimolecular layer we might illustrate the micellae with their water-layer by the following diagram. (Fig. 1. to the left of the bracket).

It goes without saying that such micellae will attract each other with their unlike poles. But more is to happen. The micellae impart to their water-layer a charge that must induce adsorption of ions from the environing fluid and the occurrence of an electric double-charge along the water-surface. The presence of the polar charges will be the cause that this electric double-layer is not of the same compactness all along the water-surface, but that a larger number of ions will accumulate at the poles than along the side-walls. This larger accumulation of ions at the poles will lower the surface tension there.

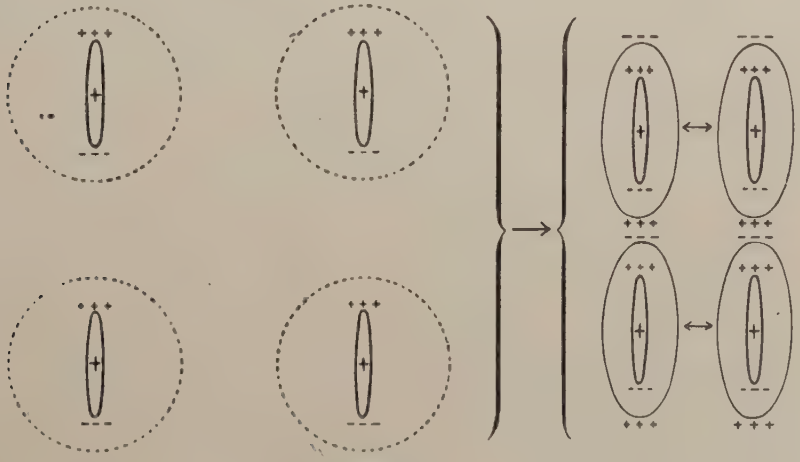

Figure 1.

The consequence is that opposite the poles the curvature of the water-envelop is more pronounced and consequently is getting ellipsoidal also round the micella.

This alters at the same time the influence which the ions exert upon each other in space. If this influence were restricted to the expelling (stabilizing) action of the like charges of ions, the micellae would in virtue of their polar charges (in case they extend in the same direction, like a tendon) dispose themselves in all directions at the same distance from each other. But there is another complicating influence at work, viz. the mutual attraction of the highly distended sidewalls of the water-envelops. Repulsion by micellar charge, attraction by the water-envelops, these two counteracting forces bring about the actual distribution of the fibrils in space. The relation between these forces is to account for the tendency of the fibrils to form bundles. That the fibril-bundles are not cylindrical but tape-like, implies that the attraction of the fibrils is more intense in the one spatial direction than in the other. This again involves that also the transverse section of the water-envelop (vertical to the long axis of the micella) has the form of an ellipse. In harmony with this the fibrils are not, as has often been represented, uniaxial, but according to SCHMIDT biaxial birefringent. If this reasoning is correct, the mutual influence of the ions in space might be figured as shown to the right of the brackets in fig 1. It appears to us that our exposition of the structure of the collagenous fibrils offers a plausible explanation of a number of physical properties they possess.

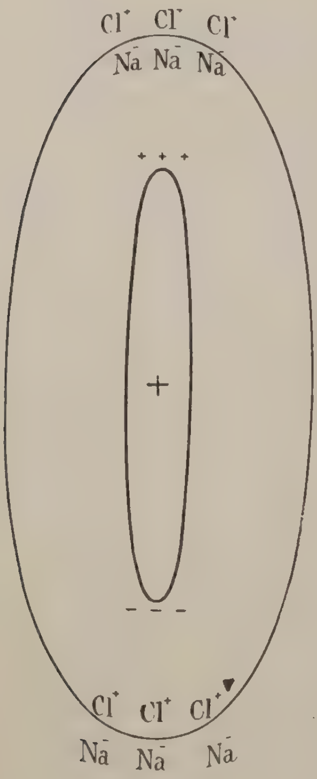

Figure 2.

1⁰ Their faculty of swelling in a faintly acid and in a faintly alkaline environment.

Given the tension elicited by the polar charges in the ellipsoidal water-envelop, that envelop will display a tendency to become spherical. This can be only when the surface tension of the sidewalls may diminish relative to the polar curvatures. Given also the fact that, as we suppose, the polar charges of the envelop are stabilized by those of the micella, an increase of the curvature of the sidewalls is possible only if from the outside ions can be transmitted to the surface, i.e. by supplying free + ly or — ly charged ions. If, however, under these circumstances the sidewalls are to be globed, this will be possible only with increase of the volume of the whole enveloping ellipsoid, i.e. with an adsorption of water (acid-, and alkali-swelling). A shortening of the long axis is hardly required here, and will take place only when at length the side charges approach those of the poles, and thereby the whole tends to be globular. Indeed, with acid-swelling a slight but distinct shortening manifests itself.

Two factors will co-operate to bring about a thickening of the tendon with acid-hydration : 1⁰ the lengthening of the short axis of the ellipsoidal water-envelops themselves ; 2⁰ the increasing expulsion of the fibrils by their like micellar charges, in proportion to the decrease of attraction of the curving water-surfaces. In agreement with this is the fact that with increasing hydration of the connective tissue (edema) the fibril-bundles are gradually discomposed. The fibril-isolating action of line and baryt-water, hitherto ascribed to "sticky matter being dissolved", might also be plausibly interpreted in an analogous way by the accumulation of ions of the water-envelops.

2° The shortening of the long axis will, however, be more pronounced, if *without wateradsorption* the particles can contract into a sphere. According to the conception put forward this would be possible only with abolishment of the polar charge, that is to say, since the polar charge of the water- envelop is secondary to that of the micellae, with abolishment of the latter. Well then, whereas the reversible hydration-swelling, as stated before, involves only a relatively insignificant shortening, we see a tendon contrast suddenly in warm *neutral* water at 60° (the temperature for denaturation of protein !). Does not this phenomenon point to a discontinuance of a tension, which may the more readily be attributed to molecular changes of the micella, if we add to what has been said, that the changes appearing in the birefringence are in this case not reversible, and that, through distention, only a part of the original "Stäbchen Doppelbrechung" can be recovered ?

3° That reversible to less than 60° shortening of a tendon caused by heat is to be explained by rounding of the ellipsoidal micellae, is a conclusion based long since on the changes occurring in the double refraction. This rounding which in the first instance we think fit to localize in the water-envelops, we might ascribe to decrease of the surface-tension of the water-envelops, which thereby are forced to intenser curving.

In the foregoing we have set forth that with our hypothesis an explanation may be given of a number of properties of collagenous fibrils. It occurs to us that by the same reasoning, when applied to other fibrils, e.g. muscle-, and nerve-fibres, whose analogous structural development from a row of micellae has been established by röntgenologic investigation, several physiological facts may be elucidated. It would be premature, however, to go into that problem now. We know quite well that our reasoning is purely hypothetical. It may be that principles of colloid-chemistry have not been stabilized enough to apply its definitions and speculations at this moment to the immensely complicated systems in living nature. It may also be that we, being laymen, have with our lack of experience misused principles, of themselves unexceptionable. We trust that, even if the theoretical speculation, to which the second half of our paper has been devoted, should be invalidated by the critique of more competent observes than ourselves, the observations of the facts recorded in the first half will at any rate be admitted to be unexceptionable. Our endeavour to refer connective tissue to the domain of colloid-chemistry will not be made in vain if it would help to convince biologists, that also outside of protoplasm there can be question of life, in which genesis, existence and dissolution succeed each other according to fixed laws.

Mathematics. — *"On a Group of Representations of the Linear Complex of Rays"*. By M. N. VAN DER BIJL. (Communicated by Prof. JAN DE VRIES).

(Communicated at the meeting of April 25, 1925).

§ 1. We have in view all the representations of a linear complex L on the points of space *for which each ray contains its own image.* These have a number of properties in common of which the most important follow here. In the first place these two:

a. A plane pencil out of L is represented on a conic k^2 through the null point of the plane of the pencil which is touched at this point by the ray that has its image in the point.

b. The image of a net of rays out of L is a cubic surface O^3 through the directrices of the net (u and v). O^3 has the image of u as double point if u belongs to L.

Proof of *a*: the image curve is plane, has a single point in the null-point, and cuts any ray of the pencil, besides, in the image point of this ray.

Proof of *b*: A plane ν through u contains of the image surface the conic k^2 that corresponds to the pencil in ν, and u itself (each point of u corresponds as image to one ray of the net). The same holds good for v, the polar line of u. If $u \equiv v$ is a complex ray, the other part of the intersection, k^2, continues to pass through the image point of u if ν turns round u. This point is, therefore, a node of O^3.

§ 2. *The singular figures.*

If ν turns in the indicated way, k^2 degenerates 5 times into a pair of lines, according to a well known property of the cubic surfaces. The vertices of complex-plane pencils with degenerate image curves form, therefore, a surface of the fifth order; for the arbitrary line v contains 5 of these vertices. As the degeneration occurs only when the corresponding plane pencil contains a singular ray, v is cut by 5 such rays; in other words: there exists a scroll of the fifth order $R^5{}_s$ of singular rays (*singular scroll*). It coincides with the surface of the fifth order mentioned above.

The nodal curve δ of this surface is the locus of the singular points (*singular curve*), for a point on δ is among others the image of 2 singular rays and, inversely, through any singular point there must pass 2 singular rays, to wit the pair into which k^2 degenerates for this point.

Let S be a point of δ, s_1 and s_2 the 2 singular rays through S. A plane ν through s_1 cuts R^5_s along a figure of the fifth order consisting of s_1 and a curve of the fourth order which cuts s_1 in the point of contact of ν and in 3 more points, including S, which belong to δ. Applying the same to s_2 we find in all 5 points (δ, ν). A point P of ν outside s_1 and s_2 cannot belong to δ, for then the complex ray SP would be singular so that the image of the plane pencil S would become a figure of the third order. In this way we have found:

There exists a singular scroll of the fifth order with a nodal quintic as singular curve (δ^5_s).

Considering these figures we may add to property I_a that k^2 is wholly defined by the 5 points of intersection of δ^5_s and the plane of the pencil; and to I_b that O^3 contains 5 singular rays, to wit the images of the points (u, R^5_s), and further the whole curve δ^5_s, because each of its points corresponds as image to the ray of the net passing through it.

§ 3. *a. Image of a point-range u.* Let v be again the associated polar line. A plane ν through u contains 2 generatrices of the image: the 2 rays that join the nullpoint (ν, v) to the points of intersection (k^2, u). Through any point of u there passes one generatrix: the image ray of that point. Accordingly we find a cubic scroll R^3_u, that has u as single, v as double directrix. To the points (R^5_s, u) there correspond 5 image rays, which are common generatrices of R^5_s and R^3_u. If u is a complex ray, R^3_u becomes a surface of CAYLEY, for in this case u is a directrix and at the same time a generatrix.

b. Image of a field of points V. The conic which represents the plane pencil through an arbitrary point or in an arbitrary plane, cuts V in 2 points. Hence the locus of the image rays is a congruence [2,2]. 16 plane pencils belong to this. In the first place the pencil in V; further 5 plane pencils with vertices in (V, δ^5_s); the remaining 10 lie in the planes in which k^2 has one of the 10 joins of these vertices as a non-singular component. Non-singular component of a degenerate k^2 can be all the chords k of δ^5_s in the plane through k and the singular ray through the point (k, R^5_s) outside δ^5_s. The congruence [2, 2] also contains the whole singular scroll; it is produced by the points of its intersection with V.

§ 4. *a.* Image of a plane curve r^n (in the plane V with nullpoint N). This intersects V along r^n and along the $2n$ rays of the plane pencil (N, V) that are the images of the points (r^n, k^2_V). It is accordingly a scroll R^{3n} with r^n as directrix and with $5n$ singular rays among its generatrices owing to the $5n$ points of intersection (r^n, R^5_s). The nodal curve of R^{3n} passes through the $2n(n-1)$ points of intersection of r^n with the generatrices through N in so far as they do not lie on k^2_V.

b. Image of a twisted curve ϱ^n. The congruence (u, v) of L has as image an O^3. To the $3n$ points (ϱ^n, O^3) correspond $3n$ rays, that rest on u and have their images on ϱ^n. The image scroll is accordingly an R^{3n}. Among the generatrices there are $5n$ singular rays corresponding to the points (ϱ^n, R^5_s)

c. Image of a scroll R^n. This is found through inversion of *b.* The image curve ϱ^x has as image an R^{3x} but also R^n completed by the null planes of the $5n$ singular points (R^n, δ^5_s) that lie on ϱ^x. Hence: $3x = n + 5n$, consequently $\varrho^x = \varrho^{2n}$.

Also directly in the following way: ϱ^x cuts a plane in as many points as there are rays common to R^n and the congruence [2,2] which represents the points of V. And for this number we find $2n$.

§ 5. *a.* Image of a surface O^n. As O^n cuts the conic k^2 of an arbitrary point in $2n$ points and also the k^2 of an arbitrary plane, the image congruence is a $[2n, 2n]$. Any generatrix of R^5_s has n points in common with O^n so that the singular scroll, counted n times, belongs to the $[2n, 2n]$. Further $5n$ plane pencils of the congruence correspond to the points (O^n, δ^5_s).

b. Image of a $[p, p]$ of L. Let this be a surface O^x. This passes through δ^5_s with p leaves because in any point of the singular curve p rays of the $[p, p]$ are represented. Now inversely the image of O^x is a $[2x, 2x]$, but also the $[p, p]$ completed by p times the congruence [5, 5] which consists of all the complex-plane pencils with nullpoints on δ^5_s. This leads to the equation $2x = p + 5p$, hence $O^x = O^{3p}$.

§ 6. A straight line u that contains 3 singular points, is a (singular) ray, as otherwise the scroll R^3_u which represents the points of u, would be of an order higher than 3. Inversely a singular ray u always contains 3 singular points for if this number were more or less the order of R^3_u would be too high or too low; in other words: u is a trisecant of δ^5_s. Hence R^5_s is *the scroll of the trisecants of* δ^5_s.

For this reason δ^5_s cannot be a rational curve. For this would have a surface of trisecants of the order 8. The singular curve is of the genus 1. This appears as follows. The nets of rays (u, v) and (w, x) out of L have as images cubic surfaces O^3_u and O^3_w, which cut each other along δ^5_s and along a ϱ^4 on which the scroll $R^2 = (u, v, w)$ is represented which is common to the two congruences [1, 1]. The generatrices of R^2 which belong to the nets, are unisecants of ϱ^4 for all the representations in question. Such a generatrix can only have its image point in common with ϱ^4 as it is not cut by any other straight line of the same kind and consequently cannot contain any other image. The other system of straight lines on R^2 consists, accordingly, of trisecants of ϱ^4 and this curve is, therefore, rational; the number of its apparent double points is 3.

Now the theory of the intersection of 2 algebraic surfaces O^m and O^n of which the intersection ϱ^{mn} consists of a ϱ^p and a ϱ^q, teaches that:

$$h_{pq} + 2h_q = q\,(m-1)\,(n-1).$$

h_{pq} is the number of straight lines through a point P which have one point in common wich ϱ^p and another with ϱ^q and h_q is the number of chords of ϱ^q through P.

We apply this to $\varrho^9 = (O^3_u, O^3_w) = \varrho^4 + \delta^5_s$. Now $h_{pq} = 10$, for although the cones (P, ϱ^4) and (P, δ^5_s) have 20 generatrices in common, 10 of them do not cut ϱ^4 and δ^5_s in different points for ϱ^4 contains the 10 points (R^2, δ^5_s). Further $q = 5$, $m = n = 3$, so that the above mentioned equation gives: $10 + 2h_q = 5(3-1)(3-1)$, consequently $h_q = 5$, hence δ^5_s is of the genus 1.

It is in accordance with this that we can pass ∞^4 cubic surfaces through δ^5_s. For the complex contains ∞^4 nets (u, v) and each of them has its own image O^3 through δ^5_s. The condition that O^3 must pass through δ^5_s is accordingly 15-fold.

§ 7. Also several properties regarding degeneration of image figures hold good for all the representations in question, e. g.: there exists a complex of the fifth order of lines u for which R^3_u degenerates into a scroll and a plane; this complex contains all the lines that rest on δ^5_s and has, therefore, this curve as locus of the cardinal points. A congruence [5, 10] belongs to it which consists of the chords of the singular curve which contains lines u for which R^3_u degenerates into a triple of planes one of which passes through u. To this congruence there belongs again the singular scroll for the generatrices of which R^3_u degenerates into 3 planes through u. As a locus of points R^5_s is the set of the vertices of all the plane pencils with degenerate image-conics k^2, the non-singular components of these pairs of lines form the above mentioned congruence [5, 10].

§ 8. The singular figures themselves can also degenerate. This will appear from a few examples which serve at the same time as a check on the above.

a. Suppose a fixed plane α and in it 2 projective pencils (F_1, α) and (F_2, α); further a fixed straight line a through F_2 outside α. A ray s of L cuts one ray t_1 of (F_1, α). Associated to this is t_2 of (F_2, α). Put $\mu = (a, t_2)$ and take $S = (\mu, s)$ as image of s. Inversely we can find the image ray of a point S through the following construction: $\mu = (a, S)$; $t_2 = (\mu, \alpha)$; t_2 gives the homologous ray t_1 of (F_1, α); $T = (\nu, t_1)$, where ν is the null-plane of S; $s = S\,T$.

b. If S is a point of a, μ is indefinite; also t_2 and, accordingly, t_1, so that T becomes any point of (ν, α) and s any ray of (S, ν). In other words: all the points of a are singular points.

The projective pencils in α produce a conic α^2 through F_1 and F_2. A point S on it defines a definite plane $\mu = (S, a)$, hence also definite lines t_2 and t_1, but $T = (\nu, t_1) \equiv S$, so that for s we may choose any ray of the plane pencil $(S, \nu) : a^2$ is another part of the locus of the singular points.

Let t_1, t_2 and μ have the usual meaning. Let β be the nullplane of F_1; this passes through A, the nullpoint of α. β contains p_1 through A, the directrix of L associated to t_1, and $p_2 = (\mu, \beta)$. Consider $S = (p_1, p_2)$, which lies in μ. The null plane ν of S passes through t_1; the point of intersection (ν, t_1) is, therefore, indefinite; consequently S is a singular point. Owing to the correspondence (1,1) between the plane pencils (p_1) and (p_2), the locus of S is a conic b^2 in β through A and $B = (a, \beta)$. Let C be the second point of intersection of AF_1 with a^2 and choose $p_1 = AC$. Then $t_1 = AC$, because AC belongs to L. Further $t_2 = F_2C$ and $p_2 = BC$, hence $S = C$. Consequently the conics a^2 and b^2 cut each other in C.

Suppose that P outside a, a^2 and b^2 is a singular point. The cones (P, a^2) and (P, b^2) have 3 generatrices in common besides PC, which of course do not lie in one plane. But according to § 6 they must nevertheless be (singular) complex-rays. Hence P is not a singular point. The locus of the singular points is $a + a^2 + b^2$, a degenerate δ^5_s; it has 5 apparent double points. For the cones (P, a^2) and (P, b^2) have three generatrices in common which rest on a^2 and b^2 in two different points; and the plane (P, a) cuts each of these cones besides along PF_2, resp. PB, along one generatrix in 2 non-coinciding points.

c. A ray of (F_1, β) cuts all the rays t_1 (in F_1); t_1, t_2 and μ are therefore, indefinite, hence also $S = (\mu, s)$. The same holds good for the rays of (A, α), for these also cut all the t_1. The scroll which has a, a^2 and b^2 as directrices, is of the third order: R^3_s. Each of the generatrices of R^3_s contains 3 singular points and is, therefore, a singular ray.

Accordingly we have found a figure of the fifth order, $R^5_s = \alpha + \beta + R^3_s$, which consists of such rays. That this is the locus of these rays appears e.g. in the following way: let s be a singular ray which does not belong to R^5_s; its intersection S with α is the image of s and SA and would, therefore, be a singular point; but α does not contain any such a point outside a^2.

As it should be $a + a^2 + b^2$ appears to be the nodal curve of $\alpha + \beta + R^3_s$.

§ 9. Another possible degeneration of δ^5_s is: 2 crossing straight lines (a and a_1) with a transversal (a_2) and a conic (b^2) which cuts the former two lines. This happens in the representation through 2 projective plane pencils if we choose them in perspective correspondence. In this case a^2 degenerates into $a_2 = F_1F_2$ and the axis of perspectivity a_1, which continues to have a point C in common with F_1A through which b^2 also passes. The singular surface (R^5_s) consists of the planes α, β and (a, a_2) and the scroll that has a, a_1 and b^2 as directrices.

We can also establish a perspective correspondence between the plane pencils (A, β) and (B, β). For this it is only necessary that we associate the directrix of L corresponding to AB as homologous ray to the ray AF_2 of the plane pencil (F_2, α).

Finally we can transform both projectivities into perspective correspondences, through which δ^5_s is transformed into a skew pentagon. Instead of b^2 we find $b_1 = AB$ and a straight line b_2 cutting b_1 and cutting a_1 in C. R^5_s has degenerated into 5 planes (a, b_1), (b_1, b_2), (b_2, a_1), (a_1, a_2) and (a_2, a).

It is easily seen that for the latter degenerations the number of apparent double points is again 5 and that R^5_s has the curve δ^5_s as nodal curve.

A very special case is $F \equiv F_1 \equiv F_2$. In this case we have 2 collocal projective plane pencils; we may also use an involution of rays. Geometrically it is easily seen that again the fixed straight line a and a conic b^2 in the null plane β of F (quite analogous to the homonymous plane of § 8 *b*), belong to the singular points, and that $a + b^2$ is completed to a δ^5_s by the rays of coincidence c_1 and c_2. It appears as above that the complex pencils in β and in $\alpha \equiv (c_1, c_2)$ consist of singular rays.

The cubic scroll R^3_s splits up into the planes (a, c_1), (a, c_2) and β.

Accordingly the null plane β of F must be considered as a double plane in the locus of singular rays and, therefore, an arbitrary point of $\delta^5_s \equiv a + b^2 + c_1 + c_2$ is again the point of issue for 2 singular rays. As b^2 appears to pass through the null points of α, β, (a, c_1) and (a, c_2), all these rays are again trisecants of δ^5_s.

§ 10. Also each of the other forms of degeneration of the singular curve has its own singular scroll, degenerate or not, which may always be derived from it through the relation: $R^5_s \equiv$ *surface of trisecants of* δ^5_s.

a. $\delta^5_s \equiv$ *rational* $\delta^4 +$ *chord* k.

The singular scroll consists of the scroll of trisecants of δ^4 and the scroll formed by the chords of δ^4 through the points of k; it is easily seen that this surface is of the third order and has k as a nodal line.

b. $\delta^5_s \equiv$ *non-rational* $\delta^4 +$ *unisecant* k.

In this case the surface R^5_s is not degenerate: it consists of the chords of δ^4 that rest on k. Besides the line k, counted double, the intersection with a plane V through k contains 3 more chords of δ^4 through 3 singular points, namely the joints of the 3 points (V, δ^4) outside k; accordingly this intersection is indeed of the fifth order.

c. $\delta^5_s \equiv \delta^3 + \delta^2$; δ^3 *and* δ^2 *have* 2 *points in common.*

R^5_s consists of the pencil in the plane of δ^2 with vertex in the point where this plane is cut by δ^3 outside δ^2, and of a scroll of the fourth order consisting of the chords of δ^3 that rest on δ^2.

This is the case with a few representations of L found by professor JAN DE VRIES, e. g.:

A ray s cuts the fixed plane α in P; let p be the polar line of P relative to a given conic a^2 in α, ϱ the plane through p and a fixed point C; $S \equiv (s, \varrho)$ is chosen as the image of s. Inversely the image ray of a point S is found by choosing that ray of the null plane of S which rests on the polar line q of Q relative to a^2, if Q is the intersection of $C\,S$ and α.

It is at once clear that the complex plane pencil (A, α) consists of singular rays and a^2 of singular points. We find further that at any point of a^2 one tangent to the cone (C, a^2) may be drawn which belongs to L and does not yield any definite point S because ϱ passes through it. Closer examination shows that this kind of singular rays forms a surface of the fourth order with nodal curve δ^3 which passes through A and through the points where a^2 is touched by the tangents through A. In this way the aforesaid is justified.